MAU34804
Lecture 17
2026-03-10

Monotonicity

Suppose that A and μ are as in the last lecture and λ is real with $\lambda > \mu$. As we saw,

$$(\lambda I - A)^{-1} = \sum_{l=0}^{\infty} \lambda^{-l-1} A^l.$$

Every term on the right hand side is a matrix with non-negative entries and at least one of them has positive entries, so all entries of $\lambda I - A$ are positive.
Similarly,

$$\frac{d}{d\lambda}(\lambda I - A)^{-1} = -\sum_{l=0}^{\infty}(l+1)\lambda^{-l-2}A^l$$

and

$$\frac{d^2}{d\lambda^2}(\lambda I - A)^{-1} = \sum_{l=0}^{\infty}(l+1)(l+2)\lambda^{-l-3}A^l,$$

so all entries of $(\lambda I - A)^{-1}$ are strictly decreasing and strictly convex.

More monotonicity

We now reformulate the preceding results on the vector equation $\mathbf{y} = (\lambda I - A)^{-1}\mathbf{x}$ in terms of solutions of the equivalent system of scalar equations

$$\lambda y_i = x_i + \sum_{j=1}^{n} a_{ij} y_j .$$

This system has a unique solution for every $\lambda > \mu$ and $x_1 > 0$, ...$x_n > 0$. For this solution each y_i is positive and is a strictly increasing function of x_j for each j and is a strictly decreasing convex function of λ.
Also, it's a strictly increasing function of a_{jk} for each j and k.

More about eigenvalues

We've already seen than there are no eigenvalues larger than μ. μ is itself an eigenvalue, so it is *an* eigenvalue of largest norm, but is it *the* eigevalue of largest norm?

Not necessarily. $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is a counterexample, since it satisfies all of our hypotheses but has eigenvalues -1 and 1.

If we strengthen our assumptions to $A \gg O$ then the answer is yes though. In fact it would be enough to assume $A^m \gg O$, but I won't prove this.

$A^m \gg 0$ would mean there is an m such that for all j and k the j, k'th entry of A^m is positive. Isn't this what we've been assuming?

No, we've been assuming that for every j and k there is an m such that the j, k'th entry of A^m is positive. The order of quantifiers matters!

In any case we're now assuming $A \gg O$, so $f_{\mathbf{w}}$ maps $\Delta_{\mathbf{w}}$ to the interior of $\Delta_{\mathbf{w}}$.

Subspace decomposition

Let $V = \{\mathbf{x} \in \mathbf{R}^n \colon \mathbf{c}^T\mathbf{x} = 0\}$ and let W be the space spanned by the single vector $\mathbf{b}$.
If $\mathbf{x} \in V \cap W$ then $\mathbf{x} = \alpha\mathbf{b}$ for some $\alpha \in \mathbf{R}$ and $0 = \mathbf{c}^T\mathbf{x} = \alpha\sum_{i=1}^n b_i c_i$. This implies $\alpha = 0$ because $\sum_{i=1}^n b_i c_i > 0$. So $V \cap W = \{\mathbf{0}\}$.
But V is of dimension $n-1$ and W is of dimension 1 so $\mathbf{R}^n = V \oplus W$.
Also $A(V) \subseteq V$, since $\mathbf{c}^T A\mathbf{x} = \mu\mathbf{c}^T\mathbf{x} = 0$ if $\mathbf{c}^T\mathbf{x} = 0$, and $A(W) \subseteq W$, since $A\alpha\mathbf{b} = \alpha\mu\mathbf{b}$.
$f_\mathbf{w} \colon \Delta_\mathbf{w} \to \Delta_\mathbf{w}$ was defined by $f_\mathbf{w}(\mathbf{p}) = \mathbf{q}$, where $q_i = \frac{\sum_{k=1}^n a_{ik}p_k}{\sum_{j=1}^n \sum_{k=1}^n w_j a_{jk} p_k}.$
We've assumed $A \gg O$ so $f_\mathbf{w}$ maps $\Delta_\mathbf{w}$ into the *interior* of $\Delta_\mathbf{w}$.
Now $f_\mathbf{c}(p) = \frac{1}{\mu}A\mathbf{p}$ for $\mathbf{p} \in \Delta_\mathbf{c}$, so $\mu^{-1}A$ maps $\Delta_\mathbf{c}$ to its interior.
$\Delta_\mathbf{c}$ belongs neither to V nor to W, but its translate

$$\Sigma = \left\{\mathbf{x} \in \mathbf{R}^n \colon \mathbf{x} + \left(\mathbf{c}^T\mathbf{b}\right)^{-1}\mathbf{b} \in \Delta_\mathbf{c}\right\}$$

is a simplex of V.
$\mu^{-1}A(\Sigma) \subseteq \Sigma$ and $\mathbf{0}$ belongs to the interior of V.

Brin's Lemma

Brin's Lemma: Suppose V is a real vector space, Σ is a simplex in V whose interior contains $\mathbf{0}$, $T\colon V \to V$ is linear and $T(\Sigma)$ lies in the interior of Σ. Then T has no eigenvalues, real or complex, with eigenvalue of absolute value 1.

Assuming this lemma, $\mu^{-1}A|_V$ has eigenvalues less than 1, so the largest eigenvalue of A is μ and this eigenvalue has algebraic and geometric multiplicity 1.

Suppose $\mathbf{x} \in V$ is an eigenvector of T with eigenvalue $\nu = e^{2\pi i\alpha}$.

Let S be the subspace of dimension 1 or 2 spanned by the real and complex parts of $\mathbf{x}$.

If α is rational then there is an infinite sequence of powers of ν which are equal to 1. If α is irrational then there is an infinite sequence of powers of ν which tends to 1.

In either case there is an increasing sequence $j_1, j_2, \ldots$ such that $\nu^{j_k} \to 1$ as $k \to \infty$.

If $\mathbf{y} \in S \cap \partial\Sigma$ then

$$T^{j_k}\mathbf{y} = \frac{\nu^{j_k} + \overline{\nu}^{j_k}}{2}\mathbf{y} + \frac{\nu^{j_k} - \overline{\nu}^{j_k}}{\nu - \overline{\nu}}T\mathbf{y} \to \mathbf{y}$$

But $T^{j_k}\mathbf{y} \in T(\Sigma)$ and $T(\Sigma)$ is closed, so $\mathbf{y} \in T(\Sigma)$. $T(\Sigma)$ lies in the interior of Σ, not the boundary, so we have a contradiction, completing the proof of the lemma.

Banach again

The arguments above are tricky to find. There's an alternate argument which needs only one trick, but a fair amount of verification.

Define the Hilbert metric on $\Delta_{\mathbf{w}}$ by

$$d_H(\mathbf{x}, \mathbf{y}) = \begin{cases} \log \frac{\max_{1 \leq i \leq n}(x_i / y_i)}{\min_{1 \leq i \leq n}(x_i / y_i)} & \text{if } \mathbf{x} \neq \mathbf{y}, \\ 0 & \text{if } \mathbf{x} = \mathbf{y}. \end{cases}$$

It is not obvious, but this is a metric, and even a complete metric. Also, the topology it defines is the same as the usual metric.

With respect to this metric $f_{\mathbf{w}}$ is not only continuous but Lipschitz continuous with constant strictly smaller than 1, so the Banach fixed point theorem applies.

Closed Leontief model

In the Leontief production model we have a production matrix $A \geq O$ such that $a_{j,k}$ is the amount of the i'th good consumed in the production of the j'th good.

Producing a bundle of goods $\mathbf{y}$ consumes a bundle $A\mathbf{y}$, so the net production is $\mathbf{y} - A\mathbf{y}$.

I'll assume that A satisfies the conditions considered earlier, i.e. that every good is needed for the production of every other good, but possibly not directly.

The following conditions will be shown to be equivalent:

- For each j there is a $\mathbf{y} > \mathbf{0}$ such that the j'th entry of $\mathbf{y} - A\mathbf{y}$ is positive.
- There is a $\mathbf{y} > \mathbf{0}$ such that $\mathbf{y} - A\mathbf{y} \gg \mathbf{0}$.
- There is a vector $\mathbf{p} > \mathbf{0}$ such that $\mathbf{p} \gg A^T\mathbf{p}$.
- Every eigenvalue of A has absolute value less than 1.

The first condition says there is an equilibrium where we produce positive quantities of any chosen good.

The second condition says there is an equilibrium where we produce positive quantities of all goods.

The third condition says that there are prices which make all production processes profitable.

Proof

The conditions we need to show are equivalent are:
- For each j there is a $\mathbf{y} > \mathbf{0}$ such that the j'th entry of $\mathbf{y} - A\mathbf{y}$ is positive.
- There is a $\mathbf{y} > \mathbf{0}$ such that $\mathbf{y} - A\mathbf{y} \gg \mathbf{0}$.
- There is a vector $\mathbf{p} > \mathbf{0}$ such that $\mathbf{p} \gg A^T\mathbf{p}$.
- Every eigenvalue of A has absolute value less than 1.

The second condition directly implies the first, and the first implies the second because we can just add the $\mathbf{y}$'s corresponding to each j.

If the fourth condition holds then $I - A$ is invertible and its inverse satisfies $(I - A)^{-1} \gg O$ so we can choose any $\mathbf{x} \gg \mathbf{0}$ and

$$\mathbf{y} = (I - A)^{-1}\mathbf{x}$$

will satisfy $\mathbf{y} \gg \mathbf{0}$, which gives us the second condition.

Also $\left(I - A^T\right)^{-1} \gg O$ so we can choose and $\mathbf{r} \gg \mathbf{0}$ and set $\mathbf{p} = \left(I - A^T\right)^{-1}\mathbf{r}$ to get the third condition.

Proof, continued

- For each j there is a $\mathbf{y} > \mathbf{0}$ such that the j'th entry of $\mathbf{y} - A\mathbf{y}$ is positive.
- There is a $\mathbf{y} > \mathbf{0}$ such that $\mathbf{y} - A\mathbf{y} \gg \mathbf{0}$.
- There is a vector $\mathbf{p} > \mathbf{0}$ such that $\mathbf{p} \gg A^T\mathbf{p}$.
- Every eigenvalue of A has absolute value less than 1.

We've already seen that for every matrix of the type considered there is a $\mu > 0$ and a $\mathbf{c} \gg \mathbf{0}$ such that $A^T\mathbf{c} = \mu\mathbf{c}$, and that this μ is larger than the absolute value of all other eigenvalues.

Then

$$\mathbf{c} \cdot (I - A)\mathbf{y} = (I - A^T)\mathbf{c} \cdot \mathbf{y} = (1 - \mu)\mathbf{c} \cdot \mathbf{y}.$$

If the second condition holds then we can choose $\mathbf{y} > \mathbf{0}$ such that $(I - A)\mathbf{y} \gg \mathbf{0}$ and so the quantity on the left is positive and the quantity on the right has the same sign as $1 - \mu$, which must therefore be positive, so $\mu < 1$ and therefore all eigenvalues have absolute value less than 1, and the fourth condition holds.

Similarly, there's a $\mathbf{b} \gg \mathbf{0}$ such that $A\mathbf{b} = \mathbf{b}$. Looking at $(I - A)^T\mathbf{p} \cdot \mathbf{b}$ shows that the third condition implies the fourth.