MAU34804

Lecture 13

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Kakutani Fixed Point Theorem

Theorem 5.4 *Suppose $X \subseteq \mathbf{R}^n$ is a non-empty compact convex subset of a Euclidean space and $\Phi \colon X \rightrightarrows X$ is non-empty valued, convex valued and has closed graph. Then Φ has a fixed point, i.e. there is an $\mathbf{x}^* \in X$ such that $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.*

X is bounded, so there is an n-simplex Δ such that $X \subseteq \Delta$.

By Proposition 3.8 there is a retraction $r \colon \mathbf{R}^n \to X$. Its restriction is a retraction $r \colon \Delta \to X$.

Define $\Psi \colon \Delta \rightrightarrows \Delta$ by $\Psi(\mathbf{x}) = \Phi(r(\mathbf{x}))$ for $\mathbf{x} \in \Delta$.

Like Φ, Ψ is non-empty valued, convex valued and has closed graph.

Let K be the simplical complex consisting of Δ and its faces.

Let $K^{(j)}$ be a sequence of simplicial complexes such that $K^{(0)} = K$, $K^{(j+1)}$ is a subdivision of $K^{(j)}$ and $\mu(K^{(j)}) \to 0$.

These could be the successive barycentric subdivisions, but don't have to be.

Φ is non-empty valued so we can choose $\psi(\mathbf{v}) \in \Psi(\mathbf{v}) = \Phi(\mathbf{v})$ for each $\mathbf{v} \in \cup_{j=0}^{\infty} \mathrm{Vert}(K^{(j)})$.

By Lemma 4.7 and Proposition 4.8 the is a unique piecewise linear $f_j \colon \Delta \to \Delta$ with $f_j(\mathbf{v}) = \psi(\mathbf{v})$ for $\mathbf{v} \in \mathrm{Vert}(K^{(j)})$ and this f_j is continuous.

Proof of Kakutani, continued

By the Brouwer Fixed Point Theorem (5.3) there is a $\mathbf{z}^{(j)} \in \Delta$ such that $f_j(\mathbf{z}^j) = \mathbf{z}^{(j)}$. This $\mathbf{z}^{(j)}$ belongs to some n-simplex $\sigma^{(j)} \in K^{(j)}$ and has barycentric coordinates $t_l^{(j)}$ with respect to the vertices $\mathbf{v}_l^{(j)}$ of $\sigma^{(j)}$. Set

$$\mathbf{y}_l^{(j)} = f_j(\mathbf{v}_l^{(j)}) = \psi(\mathbf{v}_l^{(j)}) \in \Psi(\mathbf{v}_l^{(j)}).$$

Then $\sum_{l=0}^{n} t_l^{(j)} = 1$ and

$$\sum_{l=0}^{n} t_l^{(j)}\mathbf{v}_l^{(j)} = \mathbf{z}^{(j)} = f_j(\mathbf{z}^{(j)}) = f_j\left(\sum_{l=0}^{n} t_l^{(j)}\mathbf{v}_l^{(j)}\right) = \sum_{l=0}^{n} t_l^{(j)} f\left(\mathbf{v}_l^{(j)}\right) = \sum_{l=0}^{n} t_l^{(j)}\mathbf{y}_l^{(j)}.$$

Now $(t_0^{(j)}, \ldots, t_n^{(j)}, \mathbf{v}_0^{(j)}, \ldots, \mathbf{v}_n^{(j)}, \mathbf{y}_0^{(j)}, \ldots, \mathbf{y}_n^{(j)}) \in [0,1]^{n+1} \times \Delta^{2n+2}$, which is a bounded subset of $\mathbf{R}^{2n^2+3n+1}$.
By Bolzano-Weierstrass (1.4) there is a subsequence converging to some point $(t_0^{\infty}, \ldots, t_n^{\infty}, \mathbf{v}_0^{\infty}, \mathbf{v}_n^{\infty}, \mathbf{y}_0^{\infty}, \mathbf{y}_n^{\infty})$.

Proof of Kakutani, concluded

Taking limits in the equations $\sum_{l=0}^{n} t_l^{(j)} = 1$ and $\sum_{l=0}^{n} t_l^{(j)} \mathbf{v}_l^{(j)} = \sum_{l=0}^{n} t_l^{(j)} \mathbf{y}_l^{(j)}$. gives

$$\sum_{l=0}^{n} t_l^{\infty} = 1 \qquad \sum_{l=0}^{n} t_l^{\infty} \mathbf{v}_l^{\infty} = \sum_{l=0}^{n} t_l^{\infty} \mathbf{y}_l^{\infty}$$

Because $\mu(K^{(j)}) \to 0$ the $\mathbf{v}_l^{\infty}$ are all equal. Call their common value $\mathbf{x}^*$.
We then have $\mathbf{x}^* = \sum_{l=0}^{n} t_l^{\infty} \mathbf{y}_l^{\infty}$.
Also, $(\mathbf{v}_{i,j}, \mathbf{y}_{i,j}) \in \mathrm{Graph}(\Psi)$ and $\mathrm{Graph}(\Psi)$ is closed so, taking limits,
$(\mathbf{x}^*, \mathbf{y}_{i,\infty}) \in \mathrm{Graph}(\Psi)$, i.e. $\mathbf{y}_{i,\infty} \in \Psi(\mathbf{x}^*)$.
But Ψ is convex valued, so

$$\mathbf{x}^* \in \Psi(\mathbf{x}^*).$$

Finally, note that $\Psi(\mathbf{x}^*) = \Phi(r(\mathbf{x}^*)) \subseteq X$, so $\mathbf{x}^* \in X$ and $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.

Metric spaces

Both Brouwer's and Kakutani's theorems are non-constructive. They tell you a function has a fixed point, but don't give you any way to find it.

They also don't tell you the fixed point is unique, and can't without additional hypotheses.

Another fixed point theorem, not in the notes, has more restrictive hypotheses but it tells you the fixed point is unique and gives you a way to find it.

To state the Banach fixed point theorem we need the notion of a metric. A metric on a set X is a function $d\colon X \times X \to \mathbf{R}_+$, where $\mathbf{R}_+ = [0, +\infty)$, satisfying the following conditions.

- $d(\mathbf{x}, \mathbf{y}) = 0$ if and only if $\mathbf{x} = \mathbf{y}$,
- $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$, and
- $d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{y}) + d(\mathbf{y}, \mathbf{z})$.

The last of these conditions is called the triangle inequality.

I've written points in X in bold face type, like vectors, even though X doesn't have to be a subset of $\mathbf{R}^n$ because in all our examples it will be.

Some metrics on Euclidean space

Suppose X is a subset of $\mathbf{R}^n$. For any $p \in (0, \infty)$ we can define

$$d_p(\mathbf{x}, \mathbf{y}) = \left(\sum_{j=1}^{n} |x_j - y_j|^p \right)^{1/p}.$$

This function from $X \times X$ to $\mathbf{R}_+$ satisfies the first two conditions to be a metric.
If $p \geq 1$ then it also satisfies the triangle inequality.
Of course d_2 is just the Euclidean distance, which we have been using up until now.
For our applications of the Banach fixed point theorem we will want to use d_1 instead.
The proof of the triangle inequality for d_1 is particularly simple:

$$\begin{aligned} d_1(\mathbf{x}, \mathbf{z}) = \sum_{j=1}^{n} |x_j - z_j| &\leq \sum_{j=1}^{n} (|x_j - y_j| + |y_j - z_j|) \\ &\leq \sum_{j=1}^{n} |x_j - y_j| + \sum_{j=1}^{n} |y_j - z_j| = d_1(\mathbf{x}, \mathbf{y}) + d_1(\mathbf{y}, \mathbf{z}). \end{aligned}$$

Convergence, continuity, etc.

Convergence, continuity, the Cauchy condition and other notions defined in Euclidean space using the Euclidean norm can be defined in any metric space by replacing the Euclidean distance with the metric.

For example, if d_X is a metric on X and d_Y is a metric on Y then a function $f\colon X \to Y$ is said to be continuous at $\mathbf{x} \in X$ if, for each $\epsilon > 0$ there is a $\delta > 0$ such that if for any $\mathbf{w} \in X$ with $d_X(\mathbf{w}, \mathbf{x}) < \delta$ we have $d_Y(f(\mathbf{w}), f(\mathbf{x})) < \epsilon$.

Really, we should say f is continuous with respect to the metrics d_X and d_Y, but this is usually unnecessary because usually the choice of metrics is either obvious or irrelevant.

In particular, if $1 \leq p \leq q < \infty$ then

$$d_q(\mathbf{x}, \mathbf{y}) \leq d_p(\mathbf{x}, \mathbf{y}) \leq n^{\frac{1}{p} - \frac{1}{q}} d_q(\mathbf{x}, \mathbf{y}).$$

From this it follows that if f is continuous with respect to metrics d_p and d_q then it is also continuous with respect to any other pair of metrics $d_{p'}$ and $d_{q'}$.

Similarly, whether f is uniformly continuous or Lipschitz continuous doesn't depend on the metrics, although the precise relation between δ and ϵ and the Lipschitz constant do generally depend on the metrics.

Proofs

I'm mostly going to skip proofs for metric space theorems, but the proof that continuity doesn't depend on the choice of metrics gives the general flavour of all such proofs.

Suppose $f\colon X \to Y$ is continuous with respect to the metrics d_p and d_q, i.e. that for every $\mathbf{x} \in X$ and $\epsilon > 0$ there is a $\delta > 0$ such that if $d_p(\mathbf{w}, \mathbf{x}) < \delta$ then $d_q(\mathbf{w}, \mathbf{x}) < \epsilon$.

For any $\epsilon' > 0$ set $\epsilon = n^{-\max\{0, \frac{1}{q'} - \frac{1}{q}\}}\epsilon'$. Then $\epsilon > 0$ so there is a $\delta > 0$ such that if $d_p(\mathbf{w}, \mathbf{x}) < \delta$ then $d_q(f(\mathbf{w}), f(\mathbf{x})) < \epsilon$. Let $\delta' = n^{-\max\{0, \frac{1}{p} - \frac{1}{p'}\}}$.

If $d_{p'}(\mathbf{w}, \mathbf{x}) < \delta'$ then

$$d_p(\mathbf{w}, \mathbf{x}) \leq n^{\max\{0, \frac{1}{p} - \frac{1}{p'}\}} d_{p'}(\mathbf{w}, \mathbf{x}) < \delta$$

so $d_q(f(\mathbf{w}), f(\mathbf{x})) < \epsilon$ and therefore

$$d_{q'}(f(\mathbf{w}), f(\mathbf{x})) \leq n^{\max\{0, \frac{1}{q'} - \frac{1}{q}\}} d_q(f(\mathbf{w}), f(\mathbf{x})) < \epsilon'.$$

For every $\mathbf{x} \in X$ and $\epsilon' > 0$ there's a $\delta' > 0$ such that if $d_{p'}(\mathbf{w}, \mathbf{x}) < \delta'$ then $d_{q'}(f(\mathbf{w}), f(\mathbf{x})) < \epsilon'$. In other words, f is continuous with respect to the metrics $d_{p'}$ and $d_{q'}$.

d_∞

It can be shown that

$$\lim_{p \to +\infty} d_p(\mathbf{x}, \mathbf{y}) = \max_{1 \le j \le n} |x_j - y_j|.$$

We don't need this and so won't prove it but will simply define

$$d_\infty(\mathbf{x}, \mathbf{y}) = \max_{1 \le j \le n} |x_j - y_j|.$$

This is easily seen to satisfy all the required properties for a metric and the inequalities

$$d_q(\mathbf{x}, \mathbf{y}) \le d_p(\mathbf{x}, \mathbf{y}) \le n^{\frac{1}{p} - \frac{1}{q}} d_q(\mathbf{x}, \mathbf{y})$$

continue to hold if we interpret $1/\infty$ as 0.
So convergence, continuity, the Cauchy condition, etc. hold with respect to this metric if and only if they do for any other d_p, and in particular if and only if they hold in the usual Euclidean sense.

Completeness

We say that a metric d on X is *complete* if every Cauchy sequence, with respect to d, is convergent, also with respect to d.

As discussed, when dealing with subsets of $\mathbf{R}^n$ and the metrics d_p, including d_∞, a sequence is Cauchy with respect to one of these metrics if and only if it is Cauchy with respect to any of the others, and similarly for convergence.

So for any subset X of $\mathbf{R}^n$ and any $p \in [1, \infty]$ the metric d_p on X is complete if and only if the Euclidean metric is complete, i.e. if and only if X is a closed subset.

Closedness is the same with respect to d_p or the Euclidean metric.

(Banach Fixed Point Theorem) *Suppose d is a complete metric on a non-empty set X, $f : X \to X$ is a function and $c < 1$ is such that*

$$d(f(\mathbf{w}), f(\mathbf{x})) \leq c\, d(\mathbf{w}, \mathbf{x})$$

for all $\mathbf{w}, \mathbf{x} \in X$. Then f has exactly one fixed point.

This is also called the *contraction mapping principle*.