MAU34804
Lecture 12
2026-02-17

There will be no lecture tomorrow, 18 February.

Theorem 5.3 (Brouwer Fixed Point Theorem) *Let X be a subset of a Euclidean space that is homeomorphic to the closed n-dimensional ball E^n, where*

$$E^n = \{\mathbf{x} \in \mathbf{R}^n \colon |\mathbf{x}| \leq 1\}.$$

Then any continuous function $f \colon X \to X$ mapping the set X into itself has at least one fixed point $\mathbf{x}^*$ for which $f(\mathbf{x}^*) = \mathbf{x}^*$.

I already showed that any two compact convex subsets with non-empty topological interior are homeomorphic. E^n is a compact convex subset with non-empty interior, so any compact convex subset X with non-empty topological interior satisfies the hypotheses of Brouwer's theorem.

Simple examples

$f(\mathbf{x}) = \mathbf{x}$ defines a function from E^n to E^n. Every point of E^n is a fixed point of this function, showing that the fixed point whose existence is guaranteed by Brouwer's theorem needn't be unique.

$f(\mathbf{x}) = -\mathbf{x}$ defines a function from E^n to E^n. $\mathbf{x}$ is a fixed point if and only if $-\mathbf{x} = \mathbf{x}$, which happens if and only if $\mathbf{x} = \mathbf{0}$.

More interestingly, it also defines a function from the unit sphere $\{\mathbf{x} \in \mathbf{R}^n : |\mathbf{x}| = 1\}$ to itself. Since $\mathbf{0}$ is not in the domain this function has no fixed points. This shows that the assumption that X is homeomorphic to E^n can't be removed from Brouwer's theorem.

Also, once we've proved Brouwer's theorem it shows that the sphere is not homeomorphic to the ball.

A more complicated example

For $a \neq 0$ define $f \colon E^2 \to E^2$ by

$$f(x,y) = \left(-\frac{2a(x^2+y^2) - 4x - 4ay + 2a}{(ax-2)^2 + (ay-a)^2}, \frac{a(x^2+y^2) + 4ax + 4a^2y - 4y - a^2 2}{(ax-2)^2 + (ay-a)^2} \right).$$

Setting $(\xi, \eta) = f(x,y)$ a straightforward calculation gives

$$\xi^2 + \eta^2 - 1 = 4\frac{x^2+y^2-1}{(ax-2)^2 + (ay-a)^2}$$

so $\xi^2 + \eta^2 \leq 1$ if $x^2 + y^2 \leq 1$. In other words, f really is a function from E^2 to E^2. It's straightforward to check that the point $(0,1)$ is a fixed point of f for all values of a. It's less straightforward, but still possible, to check that this is the only fixed point. This point is on the boundary, so the restriction of f to the open unit ball does not have a fixed point. This shows that we can't replace the closed unit ball with the open unit ball in Brouwer's theorem.

From special to general

Suppose X is a subset of $\mathbf{R}^n$ such that every continuous function from X to itself has a fixed point, and suppose that $\varphi \colon X \to Y$ is a homeomorphism.

For any continuous $g \colon Y \to Y$ the function $f = \varphi^{-1} \circ g \circ \varphi$ is a continuous continuous function from X to X.

So f has a fixed point $\mathbf{x}^*$.

Let $\mathbf{y}^* = \varphi(\mathbf{x}^*)$. Then

$$g(\mathbf{y}^*) = \varphi(\varphi^{-1}(g(\mathbf{y}^*))) = \varphi(\varphi^{-1}(g(\varphi(\mathbf{x}^*)))) = \varphi(f(\mathbf{x}^*)) = \varphi(\mathbf{x}^*) = \mathbf{y}^*$$

so $\mathbf{y}^*$ is a fixed point of g.

$g \colon Y \to Y$ was an arbitrary continuous function, so every continuous function from Y to itself has a fixed point.

So if one subset has the property that all continuous functions from it to itself have fixed points then any subset homeomorphic to it has the same property.

In other words, to prove Brouwer in general it suffices to prove any special case, for example the case where X is a simplex.

Constructing a Sperner labelling

Suppose f is a continuous function from an n-simplex Δ in $\mathbf{R}^n$ to itself.

For any triangulation K of Δ we construct a Sperner labelling of K from f.

Let $\mathbf{u}_0, \ldots, \mathbf{u}_n$ be some ordering of the vertices of Δ.

For each vertex $\mathbf{v}$ of K let $s_0, \ldots, s_n$ be its barycentric coordinates and let $t_0, \ldots, t_n$ be the barycentric coordinates of $\varphi(\mathbf{v})$.

Then $\sum_{i=0}^{n} s_i = 1$ and $\sum_{i=0}^{n} t_i = 1$ and so $\sum_{i=0}^{n} s_i - t_i = 0$.

Choose i from among those indices which maximise $s_i - t_i$, so $s_i - t_i \geq 0$, with equality if and only if $s_k = t_k$ for all k, in which case $\mathbf{v}$ is a fixed point of f, and let $\ell(\mathbf{v}) = \mathbf{u}_i$.

Is ℓ a Sperner labelling? What we need to check is that if $\mathbf{v}$ belongs to a face of Δ then $\ell(\mathbf{v})$ is a vertex of that face.

Suppose then that $\mathbf{v}$ belongs to the face with vertices $\mathbf{u}_j$ with $j \in J$, $J \subset \{0, \ldots, n\}$.

Then $s_j = 0$ for all $j \notin J$ and hence $s_j - t_j \leq 0$.

If $\mathbf{v}$ is not a fixed point of f then $s_i - t_i > 0$ so $i \in J$.

So we have either a fixed point or a Sperner labelling.

There is a sequence $K^{(j)}$ of subdivisions of Δ with $\lim_{j \to \infty} \mu(K^{(j)}) = 0$.

Conclusion of proof

Lemma 4.6 says that the j'th barycentric subdivision has $\lim_{j\to\infty} \mu(K^{(j)}) = 0$. We've seen that there are more efficient choices but the particular choice doesn't matter. Either some vertex of some $K^{(j)}$ is a fixed point of f, in which case we're done, or each subdivision has a Sperner labelling, and hence a fully labelled simplex.

Let $\sigma^{(j)}$ be that simplex, let $\mathbf{v}_l^{(j)}$ be the vertex labelled $\mathbf{u}_l$ in $\sigma^{(j)}$, let $s_{k,l}^{(j)}$ be the k'th barycentric coordinate of $\mathbf{v}_l^{(j)}$ and let $t_{k,l}^{(j)}$ be the k'th barycentric coordinate of $f(\mathbf{v}_l^{(j)})$. For each j the point $(\mathbf{v}_0^{(j)}, \ldots, \mathbf{v}_n^{(j)})$ belongs to the bounded set Δ^{n+1} so the sequence of these points has a convergent subsequence. Let $(\mathbf{v}_0^\infty, \ldots, \mathbf{v}_n^\infty)$ be its limit.

Then $\mathbf{v}_0^\infty = \cdots = \mathbf{v}_n^\infty$ because $\lim_{j\to\infty} \mu(K^{(j)}) = 0$.

Barycentric coordinates depend continuously on the point so $\lim s_{k,l}^{(j)} = s_k$, where s_k is the k'th barycentric coordinate of $\mathbf{v}^\infty$, and the limit is with respect to our subsequence. Similarly, $\lim t_{k,l}^{(j)} = t_k$, where t_k is the k'th barycentric coordinate of $f(\mathbf{v}^\infty)$.

Since our Sperner labeling had $s_{k,k}^{(j)} \geq t_{k,k}^{(j)}$, so $s_k \geq t_k$, and $\sum_{k=0}^n s_k = \sum_{k=0}^n t_k$ we conclude that $s_k = t_k$ for all k and hence $\mathbf{v}^\infty = f(\mathbf{v}^\infty)$, i.e. that $\mathbf{v}^\infty$ is a fixed point.

Numerical approximation

The proof of the Brouwer fixed point theorem used the multidimensional Bolzano-Weierstrass theorem to select a convergent subsequence from a bounded sequence.

This proof is therefore not constructive. Indeed it is known that there can be no constructive proof of the theorem.

What do we do if we have a continuous function and want to find an approximate fixed point then?

What is it we actually want? Do we want a point which is nearly fixed, i.e. is close to its image, or do we want a point which is close to an actual fixed point?

Depending on our application, either might be reasonable.

In practice people choose a fixed subdivision of small mesh size, hopefully not using barycentric subdivision, look for a fully labeled simplex, take its barycentre, and hope for the best.

We can easily check how close the point we found is to being a fixed point, but can't check whether it's near an actual fixed point.

Non-retractibility again

Earlier I skipped the proof of the following proposition.

Proposition 5.2 *Let Δ be an n-simplex with boundary $\partial\Delta$. Then there does not exist any continuous map $r\colon \Delta \to \partial\Delta$ with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$.*

This is an easy consequence of Brouwer's theorem.

Because of what we proved earlier we can replace Δ with set homeomorphic to Δ, like the closed unit ball E^n.

Suppose such an r existed, for E^n rather than Δ, and let $f(\mathbf{x}) = -r(\mathbf{x})$.

Let $\mathbf{x}^*$ be a fixed point of f, so that $\mathbf{x}^* = -r(\mathbf{x}^*)$.

Then $\mathbf{x}^* \in \partial E^n$ and so $\mathbf{x}^* = -\mathbf{x}^*$, but there are no points on the sphere with this property.

In the notes this proposition is used to prove Brouwer's theorem.

I haven't done that, so using Brouwer to prove the proposition is not circular.