

MAU34804

Lecture 11

2026-02-16

Announcement

There will be no lecture this Wednesday, 18 February.

Fixed point theorems

The main point of Chapter 5 is the following two theorems:

Theorem 5.3 (Brouwer Fixed Point Theorem) *Let X be a subset of a Euclidean space that is homeomorphic to the closed n -dimensional ball E^n , where*

$$E^n = \{\mathbf{x} \in \mathbf{R}^n : |\mathbf{x}| \leq 1\}.$$

Then any continuous function $f : X \rightarrow X$ mapping the set X into itself has at least one fixed point \mathbf{x}^ for which $f(\mathbf{x}^*) = \mathbf{x}^*$.*

Theorem 5.4 (Kakutani's Fixed Point Theorem) *Let X be a non-empty, compact and convex subset of n -dimensional Euclidean space \mathbf{R}^n , and let $\Phi : X \rightrightarrows X$ be a correspondence mapping X into itself. Suppose that the graph of the correspondence Φ is closed and that $\Phi(\mathbf{x})$ is non-empty and convex for all $\mathbf{x} \in X$. Then there exists a point \mathbf{x}^* of X that satisfies $\mathbf{x}^* \in \Phi(\mathbf{x}^*)$.*

Brouwer is nearly a special case of the Kakutani. More precisely, the special case of Brouwer where X is the closed ball is also a special case of Brouwer.

The general case of Brouwer follows easily from this special case.

Instead, we'll prove Kakutani using Brouwer.

Sperner's labellings

I'll give a different proof Brouwer from the notes, using less of Chapter 4, but both proofs are based Sperner's Lemma.

First a definition:

Let K be a simplicial complex which is a subdivision of some n -dimensional simplex Δ .

We define a *Sperner labelling* of the vertices of K to be a function, labelling each vertex of K with an integer between 0 and n , with the following properties:

- for each $j \in \{0, 1, \dots, n\}$, there is exactly one vertex of Δ labelled by j ,
- if a vertex \mathbf{v} of K belongs to some face of Δ , then some vertex of that face has the same label as \mathbf{v} .

This definition is slightly weird. The vertices of a simplex are a set, not a list, so they aren't labelled by integers.

We can fix this by making Δ an ordered simplex, i.e. a simplex with a chosen ordering of its vertices.

Sperner labelling's, continued

We can fix this by making Δ an ordered simplex, i.e. a simplex with a chosen ordering of its vertices.

That's a bit pointless though. We're associating vertices of K with integers and then associating integers with vertices of Δ . It would be simpler just to associate vertices of K with vertices of Δ .

That's what I'll do. From now on a Sperner labelling of K is a function ℓ from vertices of K to vertices of Δ with the property that if a vertex \mathbf{v} of K belongs to the face of Δ then $\ell(\mathbf{v})$ is one of the vertices of that face.

In particular, if \mathbf{v} is a vertex of Δ then $\ell(\mathbf{v}) = \mathbf{v}$.

This incidentally removes the need to order Δ .

Sperner's Lemma

The version of Sperner's lemma in the notes is as follows.

Lemma 5.1 (Sperner's Lemma) *Let K be a simplicial complex which is a subdivision of an n -simplex Δ . Then, for any Sperner labelling of the vertices of K , the number of n -simplices of K whose vertices are labelled by $0, 1, \dots, n$ is odd.*

Since I've redefined Sperner labelling I need to change this.

It's convenient to introduce a new term. If ℓ is a Sperner labelling of K , a subdivision of Δ , and $\sigma \in K$ then $\ell(\text{Vert}(\sigma)) \subset \text{Vert}(\Delta)$. We say that σ is *fully labelled* if $\ell(\text{Vert}(\sigma)) = \text{Vert}(\Delta)$.

Lemma 5.1 v2 *Let K be a simplicial complex which is a subdivision of an n -simplex Δ . Then, for any Sperner labelling ℓ of the vertices of K , the number of fully labelled simplices of K is odd.*

Usually we really only care that it's positive.

It's also convenient to use the terminology of congruence modulo 2. So odd, for example, is the same as congruent to 1 modulo 2.

The idea of the proof

Suppose K , Δ , and ℓ are as before, and Σ is a face of Δ . We can define a complex K_Σ consisting of all those simplices in K which are subsets of Σ . This K_Σ is a subdivision of Σ and $\text{Vert}(K_\Sigma) = \text{Vert}(K) \cap \Sigma$.

Let ℓ_Σ be the restriction of ℓ to $\text{Vert}(K_\Sigma)$. Then ℓ_Σ is a Sperner labelling of the subdivision K_Σ of the simplex Σ .

Remember the defining property of Sperner labellings is $\ell(\mathbf{v}) \in \text{Vert}(\Sigma)$ for $\mathbf{v} \in \Sigma$.

The case where Σ has dimension 0, where $\Sigma = \{\mathbf{v}\}$ for some vertex \mathbf{v} of Δ , is particularly simple.

Then $K_\Sigma = \{\Sigma, \emptyset\}$, $\text{Vert}(K_\Sigma) = \{\mathbf{v}\}$, and $\ell_\Sigma(\mathbf{v}) = \mathbf{v}$.

There is exactly one fully labelled simplex in K_Σ , which is Σ itself.

So if we can show that the number of fully labelled simplices in K_Σ is congruent modulo 2 to the number of fully labelled simplices in K for any non-empty face Σ of Δ then the lemma will follow.

We can get from any face Σ to Δ by adding one vertex at a time, so we only need to consider the case $\dim(\Delta) = \dim(\Sigma) + 1$.

The details

At this point we've reduced the proof of the lemma to the following.

Suppose K is a subdivision of Δ with a Sperner labelling ℓ and Σ is a face of Δ with $\dim(\Delta) = q + 1$, where $q = \dim(\Sigma)$.

Let K_Σ be the subdivision of Σ made from the simplices in K which are subsets of Σ .

Let ℓ_K be the Sperner labelling of K_Σ obtained by restricting ℓ to $\text{Vert}(K_\Sigma)$.

Then the number of fully labelled simplices of K is congruent modulo 2 to the number of fully labelled simplices of K_Σ .

To prove this we count the pairs of simplices σ, τ of K with the following properties:

$\sigma \subset \tau$, $\dim(\sigma) = q$, $\dim(\tau) = q + 1$, $\ell(\text{Vert}(\sigma)) = \text{Vert}(\Sigma)$.

More precisely, we count it in two different ways, first asking which τ have these properties for a given σ and then asking which σ have these properties for a given τ .

First count

We are counting the pairs of simplices σ, τ of K with the following properties: $\sigma \subset \tau$, $\dim(\sigma) = q$, $\dim(\tau) = q + 1$, $\ell(\text{Vert}(\sigma)) = \text{Vert}(\Sigma)$.

First we consider all σ which could potentially be part of such a pair, i.e. those such that $\dim(\sigma) = q$ and $\ell(\text{Vert}(\sigma)) = \text{Vert}(\Sigma)$.

Such a σ is either on the boundary of Δ or in the interior.

If it's on the boundary then it's either a subset of Σ or of one of the other faces.

If σ belonged to another face then $\ell(\text{Vert}(\sigma))$ would be a subset of the vertices of that face, because ℓ is a Sperner labelling, so $\text{Vert}(\Sigma)$ would be a subset of the vertices of that face, and so Σ would be a face of that face, which is impossible.

On the other hand, if σ belongs to Σ then σ is a fully labelled simplex in the Sperner labelling ℓ_Σ of K_Σ .

If σ lies in the interior of Δ then it is a proper face of exactly two simplices τ , both of dimension $q + 1$, and both of these choices will give us a pair as above.

So the number of pairs is the number of fully labelled simplices in K_Σ plus an even number.

Second count

We are still counting the pairs of simplices σ, τ of K with the following properties:
 $\sigma \subset \tau$, $\dim(\sigma) = q$, $\dim(\tau) = q + 1$, $\ell(\text{Vert}(\sigma)) = \text{Vert}(\Sigma)$.

Now we consider all τ which could potentially be part of such a pair, i.e. those such that $\dim(\tau) = q + 1$, with at least one q -face such that $\ell(\text{Vert}(\sigma)) = \text{Vert}(\Sigma)$.

τ has exactly one vertex \mathbf{v} which is not in σ and either $\ell(\mathbf{v}) \in \text{Vert}(\Sigma)$ or $\ell(\mathbf{v}) \notin \text{Vert}(\Sigma)$.

In the first case there is some vertex \mathbf{w} of σ with $\ell(\mathbf{w}) = \ell(\mathbf{v})$ and the simplex σ' with vertices $\{\mathbf{w}\} \cup \text{Vert}(\sigma) \setminus \{\mathbf{v}\}$ is another q -face of τ with $\ell(\text{Vert}(\sigma')) = \text{Vert}(\Sigma)$, so the pair σ', τ also contributes to our count.

No other faces of τ contribute though.

In the second case the simplex τ is fully labelled, and there is no other face σ' of τ which contributes to the count.

So the number of such pairs is the number of fully labelled simplices of K plus an even number.

Conclusion

So the number of pairs is equal to the number of fully labelled simplices of K_Σ plus an even number, and also equal to the number of fully labelled simplices of K plus an even number.

In other words, the number of fully labelled simplices of K_Σ is congruent modulo two to the number of fully labelled simplices of K .

We already saw that this statement implies Sperner's lemma.

This proof is more or less the same as the proof in the notes, but the steps have been reordered.

A proposition

The notes use Sperner's lemma and the simplicial approximation theorem, which we've skipped, to prove the following non-retractability theorem.

Proposition 5.2 *Let Δ be an n -simplex with boundary $\partial\Delta$. Then there does not exist any continuous map $r: \Delta \rightarrow \partial\Delta$ with the property that $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in \partial\Delta$.*

This proposition is then used to prove Brouwer's theorem.

I'll give a different proof of Brouwer, but the proposition is useful, so I'll prove it later as a corollary to Brouwer.