MAU34804
Lecture 10
2026-02-11

Simplicial complexes

Chapter 4 of the notes is all about simplicial complexes. We'll only need the first three sections, not the last 3.

A finite collection K of simplices in $\mathbf{R}^k$ is said to be a *simplicial complex* if the following two conditions are satisfied.

- If σ is a simplex belonging to K then every face of σ also belongs to K,
- If σ_1 and σ_2 are simplices belonging to K then either $\sigma_1 \cap \sigma_2 = \varnothing$ or else $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2.

The *dimension* of a simplicial complex K is the greatest non-negative integer n with the property that K contains an n-simplex.

The *polyhedron* of a simplicial complex K is the union of all the simplices of K.

The polyhedron $|K|$ of a simplicial complex K is a subset of a Euclidean space that is both closed and bounded. It is therefore a compact subset of that Euclidean space.

Triangulation

Do not confuse the polyhedron with the simplicial complex!
Every simplicial complex has a polyhedron but a polyhedron can come from many different simplicial complexes.
Also, simplicial complexes are finite sets of simplices. Polyhedra are sets of points, infinitely many of them if the dimension of the the complex is positive.
Finding a simplicial complex whose polyhedron is a given set is called *triangulation*, even when the dimension is not two.
Here are two different triangulations of the square $\{(x, y) \in \mathbf{R}^2 \colon 0 \leq x, y \leq 1\}$:
Letting $\mathbf{v}_0 = (0, 0)$, $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (1, 1)$, and $\mathbf{v}_3 = (0, 1)$, both triangulations contain the 0-simplices $\mathrm{Simp}(\mathbf{v}_0)$, $\mathrm{Simp}(\mathbf{v}_1)$, $\mathrm{Simp}(\mathbf{v}_2)$, and $\mathrm{Simp}(\mathbf{v}_3)$, and also the 1-simplices $\mathrm{Simp}(\mathbf{v}_0, \mathbf{v}_1)$, $\mathrm{Simp}(\mathbf{v}_1, \mathbf{v}_2)$, $\mathrm{Simp}(\mathbf{v}_2, \mathbf{v}_3)$, $\mathrm{Simp}(\mathbf{v}_3, \mathbf{v}_0)$.
One of them also contains the 1-simplex $\mathrm{Simp}(\mathbf{v}_0, \mathbf{v}_2)$ and the 2-simplices $\mathrm{Simp}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$ and $\mathrm{Simp}(\mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3)$.
The other contains the 1-simplex $\mathrm{Simp}(\mathbf{v}_1, \mathbf{v}_3)$ and the 2-simplices $\mathrm{Simp}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3)$ and $\mathrm{Simp}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

Subcomplexes

Lemma 4.1 *Let K be a simplicial complex, and let X be a subset of some Euclidean space. A function $f : |K| \to X$ is continuous on the polyhedron $|K|$ of K if and only if the restriction of f to each simplex of K is continuous on that simplex.*

Let K be a simplicial complex in $\mathbf{R}^k$. A *subcomplex* of K is a collection L of simplices belonging to K with the following property:

- If σ is a simplex belonging to L then every face of σ also belongs to L.

Proposition 4.2 *Let K be a finite collection of simplices in some Euclidean space $\mathbf{R}^k$, and let $|K|$ be the union of all the simplices in K. Then K is a simplicial complex (with polyhedron $|K|$) if and only if the following two conditions are satisfied: K contains the faces of its simplices, and every point of $|K|$ belongs to the interior of a unique simplex of K.*

Barycentre, subdivisions

Let σ be a q-simplex in $\mathbf{R}^k$ with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$. The *barycentre* of σ is defined to be the point

$$\hat{\sigma} = \frac{1}{q+1}(\mathbf{v}_0 + \mathbf{v}_1 + \cdots + \mathbf{v}_q).$$

Let σ and τ be simplices in some Euclidean space. If σ is a proper face of τ then we denote this fact by writing $\sigma < \tau$.

A simplicial complex K_1 is said to be a *subdivision* of a simplicial complex K if $|K_1| = |K|$ and each simplex of K_1 is contained in a simplex of K.

An example

We saw two triangulations of the square $\{(x, y) \in \mathbf{R}^2 : 0 \leq x, y \leq 1\}$. Here is another: $\mathbf{v}_0 = (0, 0)$, $\mathbf{v}_1 = (1, 0)$, $\mathbf{v}_2 = (1, 1)$, and $\mathbf{v}_3 = (0, 1)$, as before.
Now set $\mathbf{v}_4 = (1/2, 1/2)$.
The 0-simplices of our new complex are $\mathrm{Simp}(\mathbf{v}_0)$, $\mathrm{Simp}(\mathbf{v}_1)$, $\mathrm{Simp}(\mathbf{v}_2)$, $\mathrm{Simp}(\mathbf{v}_3)$, and $\mathrm{Simp}(\mathbf{v}_4)$.
The 1-simplices are $\mathrm{Simp}(\mathbf{v}_0, \mathbf{v}_1)$, $\mathrm{Simp}(\mathbf{v}_1, \mathbf{v}_2)$, $\mathrm{Simp}(\mathbf{v}_2, \mathbf{v}_3)$, $\mathrm{Simp}(\mathbf{v}_3, \mathbf{v}_0)$, $\mathrm{Simp}(\mathbf{v}_0, \mathbf{v}_4)$, $\mathrm{Simp}(\mathbf{v}_1, \mathbf{v}_4)$, $\mathrm{Simp}(\mathbf{v}_2, \mathbf{v}_4)$, and $\mathrm{Simp}(\mathbf{v}_3, \mathbf{v}_4)$.
The 2-simplices are $\mathrm{Simp}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_4)$, $\mathrm{Simp}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_4)$, $\mathrm{Simp}(\mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4)$, and $\mathrm{Simp}(\mathbf{v}_3, \mathbf{v}_0, \mathbf{v}_4)$.
This simplicial complex is a subdivision of the two earlier ones. You should check this!
It is a theorem, which we will not prove, that for any two simplicial complexes with the same polyhedron there is a third simplicial complex which is a subdivision of each.
This is really useful for proving that properties of polyhedra defined in terms of simplicial complexes are well defined. You just need to show that the property is preserved by subdivision.

Miscellaneous facts about polyhedra

I won't prove any of these, or use them, but they are provided for cultural awareness:
- The union of finitely many polyhedra is a polyhedron.
- The intersection of finitely many polyhedra is a polyhedron.
- The set theoretic difference of two polyhedra doesn't have to be a polyhedron but its closure is a polyhedron.

Barycentric subdivision

Let K be a simplicial complex in some Euclidean space $\mathbf{R}^k$. The *first barycentric subdivision* K' of K is defined to be the collection of simplices in $\mathbf{R}^k$ whose vertices are $\hat{\sigma}_0, \hat{\sigma}_1, \ldots, \hat{\sigma}_r$ for some sequence $\sigma_0, \sigma_1, \ldots, \sigma_r$ of simplices of K with $\sigma_0 < \sigma_1 < \cdots < \sigma_r$. Thus the set of vertices of K' is the set of all the barycentres of all the simplices of K.

Proposition 4.3 *Let K be a simplicial complex in some Euclidean space, and let K' be the first barycentric subdivision of K. Then K' is itself a simplicial complex, and $|K'| = |K|$.*

Lemma 4.4 *Let σ be a q-simplex and let τ be a face of σ. Let $\hat{\sigma}$ and $\hat{\tau}$ be the barycentres of σ and τ respectively. If all the 1-simplices (edges) of σ have length not exceeding d for some $d > 0$ then*

$$|\hat{\sigma} - \hat{\tau}| \leq \frac{qd}{q+1}.$$

Barycentric subdivision, continued

The *mesh* $\mu(K)$ of a simplicial complex K is the length of the longest edge of K.

Lemma 4.5 *Let K be a simplicial complex, and let n be the dimension of K. Let K' be the first barycentric subdivision of K. Then*

$$\mu(K') \leq \frac{n}{n+1}\mu(K).$$

Lemma 4.6 *Let K be a simplicial complex, let $K^{(j)}$ be the j'th barycentric subdivision of K for all positive integers j, and let $\mu(K^{(j)})$ be the mesh of $K^{(j)}$. Then* $\lim \mu(K^{(j)}) = 0$.

Piecewise linear functions

Let K be a simplicial complex in n-dimensional Euclidean space. A function $f\colon |K| \to \mathbf{R}^m$ mapping the polyhedron $|K|$ of K into m-dimensional Euclidean space $\mathbf{R}^m$ is said to be *piecewise linear* on each simplex of K if

$$f\left(\sum_{i=0}^{q} t_i \mathbf{v}_i\right) = \sum_{i=0}^{q} t_i f\left(\mathbf{v}_i\right)$$

for all vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of K that span a simplex of K, and for all non-negative real numbers $t_0, t_1, \ldots, t_q$ satisfying $\sum_{i=0}^{q} t_i = 1$.

Lemma 4.7 *Let K be a simplicial complex in n-dimensional Euclidean space, and let $f\colon |K| \to \mathbf{R}^m$ be a function mapping the polyhedron $|K|$ of K into m-dimensional Euclidean space $\mathbf{R}^m$ that is piecewise linear on each simplex of K. Then $f\colon |K| \to \mathbf{R}^m$ is continuous.*

Piecewise linear functions

Proposition 4.8 *Let K be a simplicial complex in n-dimensional Euclidean space and let $\alpha\colon \mathrm{Vert}(K) \to \mathbf{R}^m$ be a function mapping the set $\mathrm{Vert}(K)$ of vertices of K into m-dimensional Euclidean space $\mathbf{R}^m$. Then there exists a unique function $f: |K| \to \mathbf{R}^m$ defined on the polyhedron $|K|$ of K that is piecewise linear on each simplex of K and satisfies $f(\mathbf{v}) = \alpha(\mathbf{v})$ for all vertices $\mathbf{v}$ of K.*

Corollary 4.9 *Let K be a simplicial complex in $\mathbf{R}^n$ and let L be simplicial complexes in $\mathbf{R}^m$, where m and n are positive integers, and let $\varphi\colon \mathrm{Vert}(K) \to \mathrm{Vert}(L)$ be a function mapping vertices of K to vertices of L. Suppose that $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$ span a simplex of L for all vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ of K that span a simplex of K. Then there exists a unique continuous map $\bar{\varphi}\colon |K| \to |L|$ mapping the polyhedron $|K|$ of K into the polyhedron $|L|$ of L that is piecewise linear on each simplex of K and satisfies $\bar{\varphi}(\mathbf{v}) = \varphi(\mathbf{v})$ for all vertices $\mathbf{v}$ of K. Moreover this function maps the interior of a simplex of K spanned by vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ into the interior of the simplex of L spanned by $\varphi(\mathbf{v}_0), \varphi(\mathbf{v}_1), \ldots, \varphi(\mathbf{v}_q)$.*

Other subdivisions

The point of subdivision and piecewise linear functions is that we can approximate continuous functions arbitrarily well by piecewise linear functions with respect to some subdivision, chosen to have mesh less than whatever δ corresponds to a given ϵ.

If you only need this for proofs, barycentric subdivision is fine. If you need it for computations then barycentric subdivision is terrible.

The following subdivision works much better in practice.

Order all the vertices and choose a degree d of subdivision.

Each q-simplex, with vertices $\mathbf{v}_0, \ldots, \mathbf{v}_q$, ordered according to our ordering of all the vertices, we will have d^q q-simplices in our subdivision, one for each set of numbers $i_1, \ldots, i_q$ with $0 \leq i_1, \ldots, i_q < d$.

For any such set there is a unique permutation $\sigma \in S_q$ such that $j < k$ implies $i_{\sigma(j)} < i_{\sigma(k)}$ or $i_{\sigma(j)} = i_{\sigma(k)}$ and $\sigma(j) < \sigma(k)$. Let $\tau = \sigma^{-1}$

The corresponding simplex has vertices $\mathbf{w}_0, \ldots, \mathbf{w}_q$, where

$$\mathbf{w}_l = \mathbf{v}_0 + \sum_{j=1}^{q} \frac{i_{\tau(j)} + \rho_{j,l}}{d}(\mathbf{v}_j - \mathbf{v}_{j-1}), \quad \rho_{j,l} = \begin{cases} 1 & \text{if } \tau(j) < l \\ 0 & \text{if } \tau(j) \geq k \end{cases}$$

Comparison

Suppose we have a polyhedron of dimension 10 and want to to reduce the mesh size be a factor of 10. How many polyhedra will we need?

Using the subdivision from the previous slide, we need to take $d = 20$ and we will have $20^{10} = 10,240,000,000,000$ 10-simplices in the subdivided complex for each 10-simplex in the original complex.

Using barycentric subdivision we would need to go as far as $K^{(25)}$.

We would then have $(11!)^{25}$, or

10,687,914,006,602,778,741,054,755,827,026,072,443,740,637,942,019,231,351,047,683,352,
170,533,190,894,419,220,885,240,388,621,787,003,619,551,725,196,367,632,829,047,792,
615,604,145,356,800,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000,000
10-simplices in the subdivided complex for each 10-simplex in the original complex.