MAU34804

Lecture 9

2026-02-10

More convexity

Proposition 3.7 *Let X be a closed bounded convex subset of n-dimensional Euclidean space $\mathbf{R}^n$ whose topological interior contains the origin, let S^{n-1} be the unit sphere in $\mathbf{R}^n$, defined such that*

$$S^{n-1} = \{\mathbf{u} \in \mathbf{R}^n \colon |\mathbf{u}| = 1\},$$

and let $\lambda \colon S^{n-1} \to \mathbf{R}$ be the real-valued function on S^{n-1} defined such that

$$\lambda(\mathbf{u}) = \sup\{t \in \mathbf{R} \colon t\mathbf{u} \in X\}$$

for all $\mathbf{u} \in S^{n-1}$. Then the function $\lambda \colon S^{n-1} \to \mathbf{R}$ is continuous.

In fact λ is Lipschitz continuous. Saying the topological interior contains the origin means there's an $r > 0$ such that the ball of radius r about the origin is contained in X. Saying that X is bounded means there's an $R > 0$ such that X is contained in the ball of radius R about the origin. λ is in fact Lipschitz continuous with

$$L = \frac{R\sqrt{R^2 - r^2}}{r}.$$

A corollary

The following theorem does not appear in the notes, but follows easily from Proposition 3.7.

Suppose X and Y are closed bounded convex subsets of $\mathbf{R}^n$ with non-empty topological interior. Then there is is a homeomorphism $\varphi\colon X \to Y$.

Let $\mathbf{p}$ and $\mathbf{q}$ be in the topological interiors of X and Y.

A minor extension of the proposition shows that the function $\lambda\colon S^{n-1} \to \mathbf{R}$ defined by $\lambda(\mathbf{u}) = \sup\{t \in \mathbf{R}\colon \mathbf{p} + t\mathbf{u} \in X\}$ is continuous.

The same holds for $\lambda\colon S^{n-1} \to \mathbf{R}$ defined by $\mu(\mathbf{u}) = \sup\{t \in \mathbf{R}\colon \mathbf{q} + t\mathbf{u} \in Y\}$.

Then

$$\varphi(\mathbf{x}) = \begin{cases} \mathbf{q} & \text{if } \mathbf{x} = \mathbf{p} \\ \mathbf{q} + \frac{\mu(\mathbf{u})}{\lambda(\mathbf{u})}(\mathbf{x} - \mathbf{p}) & \text{if } \mathbf{x} \neq \mathbf{p} \end{cases}$$

is a homeomorphism, where

$$\mathbf{u} = \frac{\mathbf{x} - \mathbf{p}}{|\mathbf{x} - \mathbf{p}|} \in S^{n-1}.$$

A retraction theorem

Proposition 3.8 *Let X be a closed bounded convex subset of n-dimensional Euclidean space $\mathbf{R}^n$. Then there exists a continuous map $r\colon \mathbf{R}^n \to X$ such that $r(\mathbf{R}^n) = X$ and $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$.*

Do we need closedness of X?

Suppose there is an r as above and let $\psi(\mathbf{x}) = r(\mathbf{x}) - \mathbf{x}$. $\psi(\mathbf{x}) = \mathbf{0}$ if and only if $\mathbf{x} \in X$. But ψ is continuous and the preimage of a closed set under a continuous function is closed, so X must be closed.

Boundedness turns out not to be necessary, but it makes the proof slightly easier and will be sufficient for our purposes.

A necessary hypothesis is missing though! Can you identify it?

We need X to be non-empty!

Retraction theorem, alternate proof

Proposition 3.8 *Let X be a closed bounded convex subset of n-dimensional Euclidean space $\mathbf{R}^n$. Then there exists a continuous map $r\colon \mathbf{R}^n \to X$ such that $r(\mathbf{R}^n) = X$ and $r(\mathbf{x}) = \mathbf{x}$ for all $\mathbf{x} \in X$.*

I don't like the proof in the notes, so I'll give you a different one.

Apply the Berge minimum theorem to $\Phi\colon \mathbf{R}^n \to X$ defined by $\Phi(\mathbf{x}) = X$ for all $\mathbf{x} \in \mathbf{R}^n$ and $f\colon \mathbf{R}^n \times X \to \mathbf{R}$ defined by $f(\mathbf{x}, \mathbf{y}) = |\mathbf{x} - \mathbf{y}|$.

The theorem tells us that

$$m(\mathbf{x}) = \min\{|\mathbf{x} - \mathbf{y}| : \mathbf{y} \in X\}$$

is continuous and

$$M(\mathbf{x}) = \{\mathbf{y} \in X\colon |\mathbf{x} - \mathbf{y}| = m(\mathbf{x})\}$$

is upper hemicontinuous.

Proof, continued

$$m(\mathbf{x}) = \min\{|\mathbf{x} - \mathbf{y}| : \mathbf{y} \in X\}, \quad M(\mathbf{x}) = \{\mathbf{y} \in X \colon |\mathbf{x} - \mathbf{y}| = m(\mathbf{x})\}$$

If $\mathbf{x} \in X$ then clearly $m(\mathbf{x}) = 0$ and $M(\mathbf{x}) = \{\mathbf{x}\}$.
If $\mathbf{y}_1, \mathbf{y}_2 \in M(\mathbf{x})$ then let $\mathbf{z} = \frac{\mathbf{y}_1 + \mathbf{y}_2}{2}$. We then have

$$|\mathbf{y}_1 - \mathbf{y}_2|^2 = 2|\mathbf{x} - \mathbf{y}_1|^2 + 2|\mathbf{x} - \mathbf{y}_2|^2 - 4|\mathbf{x} - \mathbf{z}|^2$$

by the parallelogram law.
Now X is convex and $\mathbf{y}_1, \mathbf{y}_2 \in X$ and therefore $\mathbf{z} \in X$ and $|\mathbf{x} - \mathbf{z}| \geq m(\mathbf{x})$. On the other hand $|\mathbf{x} - \mathbf{y}_1| = m(\mathbf{x})$ and $|\mathbf{x} - \mathbf{y}_2| = m(\mathbf{x})$ so $|\mathbf{y}_1 - \mathbf{y}_2|^2 \leq 0$ and therefore $\mathbf{y}_1 = \mathbf{y}_2$.
In other words, for each $\mathbf{x} \in \mathbf{R}^n$ there is exactly one $\mathbf{y} \in X$ such that $\mathbf{y} \in M(\mathbf{x})$. This means there is a function $r \colon \mathbf{R}^n \to X$ such that $M(\mathbf{x}) = \{r(\mathbf{x})\}$.
M is upper hemicontinuous, so r is continuous. Also, $r(\mathbf{x}) = \mathbf{x}$ for $\mathbf{x} \in X$.
We can do better than just saying r is continuous though.

Lipschitz continuity

Suppose $\mathbf{x}_1, \mathbf{x}_2 \in \mathbf{R}^n$ and let $\mathbf{y}_1 = r(\mathbf{x}_1)$ and $\mathbf{y}_2 = r(\mathbf{x}_2)$. Then $\mathbf{y}_1, \mathbf{y}_2 \in X$. Also, for any $\mathbf{z} \in X$ we have $|\mathbf{x}_1 - \mathbf{z}| \geq |\mathbf{x}_1 - \mathbf{y}_1|$.
Consider in particular $\mathbf{z} = \mathbf{y}_1 + t(\mathbf{y}_2 - \mathbf{y}_1)$ for $t \in (0, 1)$. For

$$|\mathbf{x}_1 - \mathbf{z}|^2 = |\mathbf{x}_1 - \mathbf{y}_1|^2 + 2t(\mathbf{x}_1 - \mathbf{y}_1) \cdot (\mathbf{y}_2 - \mathbf{y}_1) + t^2|\mathbf{y}_2 - \mathbf{y}_1|^2$$

to be greater than or equal to $|\mathbf{x}_1 - \mathbf{y}_1|^2$ for all small positive t we need $(\mathbf{x}_1 - \mathbf{y}_1) \cdot (\mathbf{y}_2 - \mathbf{y}_1) \geq 0$. By symmetry we also have $(\mathbf{x}_2 - \mathbf{y}_2) \cdot (\mathbf{y}_1 - \mathbf{y}_2) \geq 0$.

$$|\mathbf{x}_1 - \mathbf{x}_2|^2 = |\mathbf{y}_2 - \mathbf{y}_1|^2 + 2(\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{y}_1 + \mathbf{y}_2) \cdot (\mathbf{y}_2 - \mathbf{y}_1) + |\mathbf{x}_1 - \mathbf{x}_2 - \mathbf{y}_1 + \mathbf{y}_2|^2.$$

The last two terms on the right hand side are non-negative so $|r(\mathbf{x}_2) - r(\mathbf{x}_1)| = |\mathbf{y}_2 - \mathbf{y}_1| \leq |\mathbf{x}_1 - \mathbf{x}_2| = |\mathbf{x}_2 - \mathbf{x}_1|$. In other words, r is Lipschitz continuous with $L = 1$.

A lemma

Lemma 3.9 *Let m be a positive integer, let F be a non-empty closed set in $\mathbf{R}^m$, and let $\mathbf{b}$ be a vector in $\mathbf{R}^m$. Then there exists an element $\mathbf{g}$ of F such that $|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$ for all $\mathbf{x} \in F$.*

If F were compact this would follow immediately from the multidimensional extreme value theorem.

Choose $\mathbf{h} \in F$ and define

$$F_0 = \{\mathbf{x} \in F \colon |\mathbf{x} - \mathbf{b}| \leq |\mathbf{h} - \mathbf{b}|\}.$$

F_0 is compact, so the theorem does apply, and there is a $\mathbf{g} \in F_0$ such that $|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$ for $\mathbf{x} \in F_0$.

In particular, $|\mathbf{h} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{h}|$.

If $\mathbf{x} \in F$ then $\mathbf{x} \in F_0$ or $\mathbf{x} \in F \setminus F_0$. In the former case $|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$. In the latter case $|\mathbf{x} - \mathbf{b}| > |\mathbf{h} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$.

A similar trick can be used to remove the boundedness assumption from Proposition 3.8.

A generalisation

Let F and E be non-empty closed convex sets in $\mathbf{R}^m$. Then there exist $\mathbf{g} \in F$ and $\mathbf{h} \in E$ such that $|\mathbf{x} - \mathbf{y}| \geq |\mathbf{f} - \mathbf{g}|$ for all $\mathbf{x} \in F$ and all $\mathbf{y} \in G$.
Lemma 3.9 is the special case $E = \{\mathbf{b}\}$.
The proof is similar, except we show that $|\mathbf{x} - \mathbf{y}|$ has a minimum on $F \times E$.

Separating hyperplanes

Theorem 3.10 *Let m be a positive integer, let X be a non-empty closed convex set in $\mathbf{R}^m$, and let $\mathbf{b}$ be a point of $\mathbf{R}^m$, where $b \notin X$. Then there exists a linear functional $\varphi\colon \mathbf{R}^m \to \mathbf{R}$ and a real number c such that $\varphi(\mathbf{x}) > c$ for all $\mathbf{x} \in X$ and $\varphi(\mathbf{b}) < c$.*

We apply the previous lemma with $F = X$.

Convexity means that not only is $|\mathbf{x} - \mathbf{b}| \geq |\mathbf{g} - \mathbf{b}|$ for all $\mathbf{x} \in F$, but also

$$|\mathbf{g} - \mathbf{b} + t(\mathbf{x} - \mathbf{g})| \geq |\mathbf{g} - \mathbf{b}|$$

for all $\mathbf{x} \in F$ and all $t \in [0, 1]$.

Considering small t gives $(\mathbf{g} - \mathbf{b}) \cdot (\mathbf{x} - \mathbf{g}) \geq 0$ or, setting $\varphi(\mathbf{x}) = (\mathbf{g} - \mathbf{b}) \cdot \mathbf{x}$ and any $c \in (\varphi(\mathbf{b}), \varphi(\mathbf{g}))$, that $\varphi(\mathbf{x}) > c$ for $\mathbf{x} \in X$ and $\varphi(\mathbf{b}) < 0$.

Without convexity the theorem would be false. Taking $b = 0$ and $X = \{-1, 1\}$ in $\mathbf{R}$ gives a counterexample.

A generalisation

Suppose that X and Y are disjoint closed convex subsets of $\mathbf{R}^m$. Then there is a linear functional $\varphi\colon \mathbf{R}^m \to \mathbf{R}$ and a real number c such that $\varphi(\mathbf{x}) > c$ for all $\mathbf{x} \in X$ and $\varphi(\mathbf{y}) < c$ for all $\mathbf{y} \in Y$.

Theorem 3.10 is the special case $Y = \{\mathbf{b}\}$.

The proof is similar, but we need to use the generalisation of Lemma 3.9 instead of Lemma 3.9.

Supporting hyperplanes

Theorem 3.11 (Supporting Hyperplane Theorem) *Let m be a positive integer, let X be a non-empty closed convex set in $\mathbf{R}^m$, and let $\mathbf{b}$ be a point of $\mathbf{R}^m$ that belongs to the boundary of the closed convex set X. Then there exists a linear functional $\varphi\colon \mathbf{R}^m \to \mathbf{R}$ and a real number c such that $\varphi(\mathbf{x}) \geq c$ for all $\mathbf{x} \in X$ and $\varphi(\mathbf{b}) = c$.*

The proof in the notes uses Bolzano-Weierstrass, so is non-constructive.

There are more constructive proofs, possibly, depending on what you mean by being given a convex set.