MAU34804

Lecture 8

2026-02-09

A logical gap in the notes

A *q-simplex* in $\mathbf{R}^k$ is defined to be a set of the form

$$\mathrm{Simp}\left(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\right) = \left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j \colon 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0$, $\mathbf{v}_1$, ..., $\mathbf{v}_q$ are affinely independent points of $\mathbf{R}^k$. These points are referred to as the *vertices* of the simplex.
Note that the simplex is the subset of $\mathbf{R}^k$, not the list of vertices.
$\mathrm{Simp}\left(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_q\right) = \mathrm{Simp}\left(\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \ldots, \mathbf{v}_q\right)$, for example.
Strictly speaking then, this is not a valid definition. Vertices are defined in terms of the list, not the set. To make it valid one must show that if

$$\mathrm{Simp}\left(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q\right) = \mathrm{Simp}\left(\mathbf{w}_0, \mathbf{w}_1, \ldots, \mathbf{w}_r\right)$$

then

$$\left\{\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_q\right\} = \left\{\mathbf{w}_0, \mathbf{w}_1, \cdots, \mathbf{w}_r\right\}.$$

Closing the gap

For any subset S of a Euclidean space we say that $\mathbf{y} \in S$ is *non-extreme* if there are $\mathbf{x} \in S$ and $\mathbf{z} \in S$ and $\lambda \in [0, 1]$ such that $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are all distinct and $\mathbf{y} = \lambda\mathbf{x} + (1 - \lambda)\mathbf{z}$. We say that $\mathbf{y} \in S$ is *extreme* if it is not non-extreme.

If we can show that if the set of extreme points of $\mathrm{Simp}\,(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ is $\{\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_q\}$ then the definition given in the notes is okay.

Suppose that $\mathbf{y} \in \mathrm{Simp}\,(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$, i.e. that we can write it as $\mathbf{y} = \sum_{j=0}^{q} t_j \mathbf{v}_j$ with $0 \leq t_j \leq 1$ for all j and $\sum_{j=0}^{q} t_j = 1$.

Either some $t_i = 1$, in which case all the other t's are zero and $\mathbf{y} = \mathbf{v}_i$, or all $t_i < 1$, in which case $0 < t_i < 1$ for some i, since otherwise we would have $\sum_{j=0}^{q} t_j = 0$.

Considering the case where $0 < t_i < 1$ for some i, we can take $\mathbf{x} = \mathbf{v}_i$, $\lambda = t_i$ and $\mathbf{z} = \sum_{j \neq i} \frac{t_j}{1-\lambda} \mathbf{v}_j$.

It's easy to check that $\mathbf{y} \in \mathrm{Simp}\,(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$, $\lambda \in [0, 1]$, and $\mathbf{z} \in \mathrm{Simp}\,(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$, so $\mathbf{y}$ is a non-extreme point.

So every point is either one of the $\mathbf{v}$'s or is non-extreme. In other words, the extreme points are precisely the $\mathbf{v}$'s, so we're done.

Or are we?

Closing the gap in closing the gap

Is is possible that $\mathbf{y}$ satisfies both conditions? No, but this is less obvious than it looks. We have to exclude the possibility that $\mathbf{y} = \sum_{j=0}^{q} t'_j \mathbf{v}_j$ and $\mathbf{y} = \sum_{j=0}^{q} t''_j \mathbf{v}_j$ where $t'_{i'} = 1$ for some i' and $0 < t''_{i''} < 1$ for some i'' for some i''. We've only shown this is impossible if the $t'_j = t''_j$ for all j.

What we need is a lemma telling us that if $\mathbf{v}_0$, $\mathbf{v}_1$, ..., $\mathbf{v}_q$ are affinely independent in $\mathbf{R}^k$ then for any $\mathbf{x} \in \mathbf{R}^k$ there is at most one set of numbers $t_0, t_1, \ldots, t_q$ such that $\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x}$ and $\sum_{j=0}^{q} t_j = 1$.

A non-constructive proof is as follows. Suppose there were two such sets of t's, call them $t'_0, t'_1, \ldots, t'_q$ and $t''_0, t''_1, \ldots, t''_q$, and set $s_j = t''_j - t'_j$. Then $\sum_{j=0}^{q} s_j \mathbf{v}_j = \mathbf{0}$ and $\sum_{j=0}^{q} s_j = \mathbf{0}$, contradicting affine independence.

Now we're done.

Any time a hypothesis is left unused you should be suspicious of your proof!

Convex polytopes

A compact convex polytope in $\mathbf{R}^k$ is a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j \colon 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0$, $\mathbf{v}_1$, ..., $\mathbf{v}_q$ are points of $\mathbf{R}^k$.

Here I've dropped the assumption that the $\mathbf{v}$'s are affinely independent, so every simplex is a compact convex polytope, but not every compact convex polytope is necessarily a simplex.

It would be wrong to define the vertices of a compact convex polytope to be the $\mathbf{v}$'s.

It would be okay to define the vertices to be the extreme points of the polytope.

To see why, consider $\mathbf{v}_0 = (0, 1, 2)$, $\mathbf{v}_1 = (0, 2, 1)$, $\mathbf{v}_2 = (1, 0, 2)$, $\mathbf{v}_3 = (1, 1, 1)$, $\mathbf{v}_4 = (1, 2, 0)$, $\mathbf{v}_5 = (2, 0, 1)$, and $\mathbf{v}_6 = (2, 1, 0)$.

The set $\{\mathbf{v}_0, \ldots, \mathbf{v}_6\}$ determines a convex polytope in $\mathbf{R}^3$ which is not a simplex.

The same compact convex polytope is determined by the set with $\mathbf{v}_3$ removed.

$\mathbf{v}_3$ can't both be a vertex and not be a vertex of this compact convex polytope.

A constructive proof of the lemma.

If $\mathbf{v}_0$, $\mathbf{v}_1$, ..., $\mathbf{v}_q$ are affinely independent in $\mathbf{R}^k$ then for any $\mathbf{x} \in \mathbf{R}^k$ there is at most one set of numbers $t_0, t_1, \ldots, t_q$ such that $\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x}$ and $\sum_{j=0}^{q} t_j = 1$.

Our previous proof tells us this is true, but gives no way of checking whether such t's exist, or finding them if they do.

Let

$$V = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_{1,0} & v_{1,1} & \cdots & v_{1,q} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k,0} & v_{k,1} & \cdots & v_{k,q} \end{pmatrix}, \quad T = \begin{pmatrix} t_0 \\ t_1 \\ \vdots \\ t_q \end{pmatrix}, \quad X = \begin{pmatrix} 1 \\ x_1 \\ \vdots \\ x_k \end{pmatrix}.$$

If $\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x}$ and $\sum_{j=0}^{q} t_j = 1$ then $VT = X$.

In that case $V^T VT = V^T X$. But affine independence means $(V^T V)$ is invertible, so $T = \left(V^T V\right)^{-1} V^T X$.

This determines T, and hence the t's, uniquely, so there is at most one solution.

There might not be any, but we can now test this. Compute T according to the formula and check whether $VT = X$.

Checking simplex membership

A *q-simplex* in $\mathbf{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j \colon 0 \le t_j \le 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0$, $\mathbf{v}_1$, ..., $\mathbf{v}_q$ are affinely independent points of $\mathbf{R}^k$.

This definition isn't well suited to checking whether a point belongs to a simplex.

Now we can give a more direct test: To check if $\mathbf{x}$ belongs to this simplex compute $T = (V^T V)^{-1} V^T X$. If there are t's with $\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x}$ and $\sum_{j=0}^{q} t_j = 1$ then they're the entries of T, so just check whether they work.

That means checking whether $t_j \ge 0$ for all j and checking whether $VT = X$.

We can skip checking $t_j \le 1$. That condition is redundant.

Is checking $VT = X$ redundant? $VT = V(V^T V)^{-1} V^T X = V V^{-1} V^{-T} V^T X = X$.

Is this correct?

Faces

Definition Let σ and τ be simplices in $\mathbf{R}^k$. We say that τ is a *face* of σ if the set of vertices of τ is a subset of the set of vertices of σ. A face of σ is said to be a *proper face* if it is not equal to σ itself. An r-dimensional face of σ is referred to as an *r-face* of σ. A 1-dimensional face of σ is referred to as an *edge* of σ.

This definition is okay only because we've already seen that a simplex uniquely determines its set of vertices, and vice versa.

The faces of a simplex are subsets of the simplex and the proper faces are proper subsets.

This is not part of the definition; it is a consequence of the definition.

Question: Is the empty set a simplex? If so, what is its dimension?

Answer: according to the definition, yes, it's a simplex of dimension -1, and is a proper face of any other simplex.

Barycentric coordinates, interior

Let σ be a q-simplex in $\mathbf{R}^k$ with vertices $\mathbf{v}_0$, $\mathbf{v}_1$, ..., $\mathbf{v}_q$, and let $\mathbf{x} \in \sigma$. The *barycentric coordinates* of the point $\mathbf{x}$ (with respect to the vertices $\mathbf{v}_0$, $\mathbf{v}_1$, ..., $\mathbf{v}_q$) are the unique real numbers t_0, t_1, ..., t_q for which

$$\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{x} \text{ and } \sum_{j=0}^{q} t_j = 1.$$

The barycentric coordinates t_0, t_1, ..., t_q of a point of a q-simplex satisfy the inequalities $0 \leq t_j \leq 1$ for $j = 0, 1, \ldots, q$.

The *interior* of a simplex σ is defined to be the set consisting of all points of σ that do not belong to any proper face of σ.

Note that the definition of barycentric coordinates depends on the order of the vertices. If we reorder the vertices we have to reorder the coordinates in the same way.

It's convenient to introduce the notion of an ordered simplex, i.e. a simplex plus an ordering of its vertices.

We've already seen how to calculate the barycentric coordinates.

This means we also have a test for whether a point lies in the interior.

Another set of coordinates

There is another convenient coordinate system on an ordered simplex.
Let

$$r_j = \sum_{i<j} t_i, \quad t_j = \begin{cases} r_1 & \text{if } j = 0, \\ r_{j+1} - r_j & \text{if } 0 < j < q, \\ 1 - r_q & \text{if } j = q. \end{cases}$$

Then $0 \le r_1 \le r_2 \le \cdots \le r_q \le 1$.
Also

$$\sum_{j=0}^{q} t_j \mathbf{v}_j = \mathbf{v}_0 + \sum_{j=1}^{q} r_j(\mathbf{v}_j - \mathbf{v}_{j-1}).$$

This point lies in the interior if and only if $0 < r_1 < r_2 < \cdots < r_q < 1$.
This coordinate system is somewhat simpler, since a q simplex has q coordinates and we don't have to worry about the extra condition $\sum_{j=0}^{q} t_j = 1$.
It's badly behaved under reordering the vertices though.

Two lemmas and one warning about interiors

The following two useful lemmas are proved in the notes.

Lemma 3.2 *Let σ be a q-simplex in some Euclidean space with vertices $\mathbf{v}_0$, $\mathbf{v}_1$, ..., $\mathbf{v}_q$. Let $\mathbf{x}$ be a point of σ, and let t_0, t_1, ..., t_q be the barycentric coordinates of the point $\mathbf{x}$ with respect to $\mathbf{v}_0$, $\mathbf{v}_1$, ..., $\mathbf{v}_q$, so that $t_j \geq 0$ for $j = 0, 1, \ldots, q$, $\mathbf{x} = \sum_{j=0}^{q} t_j \mathbf{v}_j$, and $\sum_{j=0}^{q} t_j = 1$. Then the point $\mathbf{x}$ belongs to the interior of σ if and only if $t_j > 0$ for $j = 0, 1, \ldots, q$.*

Lemma 3.3 *Any point of a simplex belongs to the interior of a unique face of that simplex.*

More precisely it belongs to the simplex whose vertices correspond to the positve barycentric coordinates.

Note that the word interior, without any adjective in front, does not refer to the topological interior, which always has the word topological in front in the notes.

Convexity

A subset X of n-dimensional Euclidean space $\mathbf{R}^n$ is said to be convex if $(1-t)\mathbf{u} + t\mathbf{v} \in X$ for all points $\mathbf{u}$ and $\mathbf{v}$ of X and for all real numbers t satisfying $0 \leq t \leq 1$.

Lemma 3.4 *A simplex in a Euclidean space is a convex subset of that Euclidean space.*

Lemma 3.5 *Let X be a convex subset of n-dimensional Euclidean space $\mathbf{R}^n$, and let σ be a simplex contained in $\mathbf{R}^n$. Suppose that the vertices of σ belong to X. Then $\sigma \subset X$.*

Lemma 3.6 *Let X be a convex set in n-dimensional Euclidean space $\mathbf{R}^n$, and let $\mathbf{x} = (1-t)\mathbf{u} + t\mathbf{v}$ where $\mathbf{u}, \mathbf{v} \in X$ and $0 < t < 1$. Suppose that either $\mathbf{u}$ or $\mathbf{v}$ belongs to the topological interior of X. Then $\mathbf{x}$ belongs to the topological interior of X.*

Various kinds of continuity

Suppose X and Y are subsets of $\mathbf{R}^n$ and $\mathbf{R}^n$ and $f\colon X \to Y$ is a function.

f is continuous if for all $\mathbf{x} \in X$ and all $\epsilon > 0$ there is a $\delta > 0$ such that if $\mathbf{w} \in X$ and $|\mathbf{w} - \mathbf{x}| < \delta$ then $|f(\mathbf{w}) - f(\mathbf{x})| < \epsilon$.

f is *uniformly continuous* if for all $\epsilon > 0$ there is a $\delta > 0$ such that if $\mathbf{x} \in X$, $\mathbf{w} \in X$ and $|\mathbf{w} - \mathbf{x}| < \delta$ then $|f(\mathbf{w}) - f(\mathbf{x})| < \epsilon$.

The only difference is now the δ is only allowed to depend on the ϵ, not on the $\mathbf{x}$.

Uniform continuity implies continuity.

If X is compact then continuity implies uniform continuity.

f is *Lipschitz continuous* there is an L such that if $\mathbf{x} \in X$ and $\mathbf{w} \in X$ then $|f(\mathbf{w}) - f(\mathbf{x})| \leq L|\mathbf{w} - \mathbf{x}|$.

For example, the identity function is always Lipschitz continuous with $L = 1$.

Lipschitz continuity implies uniform continuity, since we can take $\delta = \epsilon/L$.

Uniform continuity does not imply Lipschitz continuity.

All of these notions make sense for metric spaces.