MAU34804

Lecture 7

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Simplices and Convexity

Chapter 3 of the notes deals with simplices and convexity.

We've seen a few examples of simplices but I haven't defined the term yet. We haven't seen convexity yet in this module.

Points $\mathbf{v}_0$, $\mathbf{v}_1$, ..., $\mathbf{v}_q$ in some Euclidean space $\mathbf{R}^k$ are said to be *affinely independent* (or *geometrically independent*) if the only solution of the linear system

$$\begin{cases} \sum_{j=0}^{q} s_j \mathbf{v}_j = \mathbf{0}, \\ \sum_{j=0}^{q} s_j = 0 \end{cases}$$

is the trivial solution $s_0 = s_1 = \cdots = s_q = 0$.

The term geometrically independent will never appear again.

If $v_{i,j}$ denotes the i'th coordinate of $\mathbf{v}_j$ then

$$\sum_{j=0}^{q} s_j \mathbf{v}_j = 0$$

means $\sum_{j=0}^{q} v_{i,j} s_j = 0$ for each i.

Affine independence and matrices

The conditions

$$\sum_{j=0}^{q} s_j = 0, \quad \sum_{j=0}^{q} v_{i,j} s_j = 0$$

are equivalent to the single matrix equation

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_{1,0} & v_{1,1} & \cdots & v_{1,q} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k,0} & v_{k,1} & \cdots & v_{k,q} \end{pmatrix} \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_q \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Saying that the only solution is the trivial solution $s_0 = s_1 = \cdots = s_q = 0$ means that the null space of the matrix is trivial.
This is why I put the indices in the order I did, i.e. in $v_{i,j}$ the i indicates the component and the index j indicates the vector.

Affine independence and matrices, continued

If we ignore the zeroeth row then the columns of

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_{1,0} & v_{1,1} & \cdots & v_{1,q} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k,0} & v_{k,1} & \cdots & v_{k,q} \end{pmatrix}$$

are just the $\mathbf{v}$'s, written as column vectors in the usual way.
It's often useful to think of the $\mathbf{v}$'s as living in a hyperplane of $\mathbf{R}^{k+1}$ with initial coordinate 1.
The condition that the null space is trivial is the same as the condition that the rank is equal to the number of rows, which is $k+1$.
The rank is always less than or equal to the number of columns, which is $q+1$ in this case, so $q \leq k$ is a necessary condition for the vectors to be affinely independent.
In other words, any set of more than $k+1$ vectors in $\mathbf{R}^k$ is affinely dependent.

Affine independence and Gaussian elimination

A necessary and sufficient condition for a matrix to have trivial null space is that its row echelon form has no zeroes on the main diagonal.

The row echelon form is computed using Gaussian elimination.

Example: Are the following vectors in $\mathbf{R}^3$ affinely independent?

$$\mathbf{v}_0 = (0,1,1), \quad \mathbf{v}_1 = (1,1,2), \quad \mathbf{v}_2 = (1,2,3), \quad \mathbf{v}_3 = (2,3,5)$$

Gaussian elimination gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so no.

Affine independence and Gaussian elimination, continued

Another necessary and sufficient condition for a matrix to have trivial null space is that its *column* echelon form has no zero columns.

The column echelon form is computed using Gaussian elimination, but with the roles of rows and columns switched.

Consider

$$\mathbf{v}_0 = (0,1,1), \quad \mathbf{v}_1 = (1,1,2), \quad \mathbf{v}_2 = (1,2,3), \quad \mathbf{v}_3 = (2,3,5)$$

again.

Column elimination gives

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 1 & 1 & 2 & 3 \\ 1 & 2 & 3 & 5 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

so again no.

Another condition for affine independence

Lemma 3.1 *Let* $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ *be points of Euclidean space* $\mathbf{R}^k$ *of dimension* k*. Then the points* $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ *are affinely independent if and only if the displacement vectors* $\mathbf{v}_1 - \mathbf{v}_0, \mathbf{v}_2 - \mathbf{v}_0, \ldots, \mathbf{v}_q - \mathbf{v}_0$ *are linearly independent.*

Here's a matrix proof: Multiplication from the left or right by invertible matrices doesn't change the dimension of the null space and

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -v_{1,0} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -v_{k,0} & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_{1,0} & v_{1,1} & \cdots & v_{1,q} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k,0} & v_{k,1} & \cdots & v_{k,q} \end{pmatrix} \begin{pmatrix} 1 & -1 & \cdots & -1 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & v_{1,1} - v_{1,0} & \cdots & v_{1,q} - v_{1,0} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & v_{k,1} - v_{k,0} & \cdots & v_{k,q} - v_{k,0} \end{pmatrix}.$$

The matrix on the right hand side has linearly independent columns if and only if the matrix obtained by deleting its initial row and column has linearly independent columns.
The proof in the notes is essentially this, but written in terms of vectors, not matrices.
Also, the new matrix is represents the first step in row and column elimination of the old one.

Yet another condition for affine independence

Let $\mathbf{v}_0$, $\mathbf{v}_1$, ..., $\mathbf{v}_q$ be points of Euclidean space $\mathbf{R}^k$ of dimension k. Then the points $\mathbf{v}_0$, $\mathbf{v}_1$, ..., $\mathbf{v}_q$ are affinely independent if and only if the displacement vectors $\mathbf{v}_1 - \mathbf{v}_0$, $\mathbf{v}_2 - \mathbf{v}_1$, ..., $\mathbf{v}_q - \mathbf{v}_{q-1}$ are linearly independent.

This does not appear in the notes, but it can be proved by the same method, just using the identity

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ -v_{1,0} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -v_{k,0} & 0 & \cdots & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_{1,0} & v_{1,1} & \cdots & v_{1,q} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k,0} & v_{k,1} & \cdots & v_{k,q} \end{pmatrix} \begin{pmatrix} 1 & -1 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -1 \\ 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & v_{1,1} - v_{1,0} & \cdots & v_{1,q} - v_{1,q-1} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & v_{k,1} - v_{k,0} & \cdots & v_{k,q} - v_{k,q-1} \end{pmatrix}.$$

Note that if a set of $q+1$ vectors are affinely independent then they are distinct. The converse is true for $q < 2$, but not for $q \geq 2$, since $\mathbf{v}_2 = \frac{\mathbf{v}_0 + \mathbf{v}_1}{2}$ is a counterexample for any distinct $\mathbf{v}_0$ and $\mathbf{v}_1$.

This is the last one, I promise!

The null space of a matrix A is the same as the null space of the product $A^T A$. One direction is obvious. For the other, if $\mathbf{x}$ belongs to the null space of $A^T A$ then $(A^T A)\mathbf{x} = \mathbf{0}$ so $(A\mathbf{x})^T(A\mathbf{x}) = \mathbf{x}^T(A^T A\mathbf{x}) = 0$ and so $A\mathbf{x} = \mathbf{0}$.

A square matrix has trivial null space if and only if it is invertible.

So $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q$ are affinely independent if and only if the following matrix is invertible:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_{1,0} & v_{1,1} & \cdots & v_{1,q} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k,0} & v_{k,1} & \cdots & v_{k,q} \end{pmatrix}^T \begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_{1,0} & v_{1,1} & \cdots & v_{1,q} \\ \vdots & \vdots & \ddots & \vdots \\ v_{k,0} & v_{k,1} & \cdots & v_{k,q} \end{pmatrix}.$$

We will meet this matrix again.

Simplices

A *q-simplex* in $\mathbf{R}^k$ is defined to be a set of the form

$$\left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j \colon 0 \leq t_j \leq 1 \text{ for } j = 0, 1, \ldots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0$, $\mathbf{v}_1$, ..., $\mathbf{v}_q$ are affinely independent points of $\mathbf{R}^k$. These points are referred to as the *vertices* of the simplex. The non-negative integer q is referred to as the *dimension* of the simplex. (Thus a simplex of dimension q has $q + 1$ vertices.)
Later we'll also define the term *complex*. The plural of simplex is simplices. The plural of complex is complexes.
English is a very silly language.

A logical gap in the notes

A *q-simplex* in $\mathbf{R}^k$ is defined to be a set of the form

$$\operatorname{Simp}(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_q) = \left\{ \sum_{j=0}^{q} t_j \mathbf{v}_j \colon 0 \le t_j \le 1 \text{ for } j = 0, 1, \dots, q \text{ and } \sum_{j=0}^{q} t_j = 1 \right\},$$

where $\mathbf{v}_0$, $\mathbf{v}_1$, ..., $\mathbf{v}_q$ are affinely independent points of $\mathbf{R}^k$.These points are referred to as the *vertices* of the simplex.

Note that the simplex is the subset of $\mathbf{R}^k$, not the list of vertices.

$\operatorname{Simp}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_q) = \operatorname{Simp}(\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \dots, \mathbf{v}_q)$, for example.

Strictly speaking then, this is not a valid definition. Vertices are defined in terms of the list, not the set. To make it valid one must show that if

$$\operatorname{Simp}(\mathbf{v}_0, \mathbf{v}_1, \dots, \mathbf{v}_q) = \operatorname{Simp}(\mathbf{w}_0, \mathbf{w}_1, \dots, \mathbf{w}_r)$$

then

$$\{\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_q\} = \{\mathbf{w}_0, \mathbf{w}_1, \cdots, \mathbf{w}_r\}.$$

Closing the gap

For any subset S of a Euclidean space we say that $\mathbf{y} \in S$ is *non-extreme* if there are $\mathbf{x} \in S$ and $\mathbf{z} \in S$ and $\lambda \in [0, 1]$ such that $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$ are all distinct and $\mathbf{y} = \lambda\mathbf{x} + (1 - \lambda)\mathbf{z}$. We say that $\mathbf{y} \in S$ is *extreme* if it is not non-extreme.

If we can show that if the set of extreme points of $\mathrm{Simp}\,(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$ is $\{\mathbf{v}_0, \mathbf{v}_1, \cdots, \mathbf{v}_q\}$ then the definition given in the notes is okay.

Suppose that $\mathbf{y} \in \mathrm{Simp}\,(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$, i.e. that we can write it as $\mathbf{y} = \sum_{j=0}^{q} t_j \mathbf{v}_j$ with $0 \leq t_j \leq 1$ for all j and $\sum_{j=0}^{q} t_j = 1$.

Either some $t_i = 1$, in which case all the other t's are zero and $\mathbf{y} = \mathbf{v}_i$, or all $t_i < 1$, in which case $0 < t_i < 1$ for some i, since otherwise we would have $\sum_{j=0}^{q} t_j = 0$.

Considering the case where $0 < t_i < 1$ for some i, we can take $\mathbf{x} = \mathbf{v}_i$, $\lambda = t_i$ and $\mathbf{z} = \sum_{j \neq i} \frac{t_j}{1-\lambda} \mathbf{v}_j$.

It's easy to check that $\mathbf{y} \in \mathrm{Simp}\,(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$, $\lambda \in [0, 1]$, and $\mathbf{z} \in \mathrm{Simp}\,(\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_q)$, so $\mathbf{y}$ is a non-extreme point.

So every point is either one of the $\mathbf{v}$'s or is non-extreme. In other words, the extreme points are precisely the $\mathbf{v}$'s, so we're done.

Or are we?