MAU34804

Lecture 6

2026-02-03

A framework for (simplified) game theory

Consider a two person turn based game where neither players gets a second turn.
Player 1 chooses a move from some set X of possible moves. Player 2 chooses a response. The allowed responses might or might not depend on Player 1's move.
Let Y be the set of moves which are a legal response to some move in X.
There is a relation L between X and Y, i.e. a subset of $X \times Y$, consisting of those pairs (x, y) for which y is a legal response to x.
Equivalently there is a correspondence $\Phi\colon X \rightrightarrows Y$ mapping each move x to the set $\Phi(x)$ of allowed responses.
After both players have moved Player 2 gets a payout $f(x, y)$ depending on Player 1's move x and their response y. We can represent this as a function $f\colon X \times Y \to \mathbf{R}$.
Really, only the restriction of f to L matters.
We're not assuming this is a zero sum game. Player 1's payout is not necessarily $-f$.
As Player 2 you are interested in finding the optimal response(s) $M(x)$ for each $x \in X$, in the sense of maximising f over $\Phi(x)$.
You're also interested in the maximum payout $m(x)$.
As Player 1 you are also interested in these, especially if the game *is* a zero-sum game.

Optimal play

The usual definition of optimal play for Player 2 is the set of moves maximising their payout, given that Player 1 has chosen move x, which is just $M(x)$.

In some cases it's debatable whether this is a sensible definition.

The word payout doesn't imply nonnegativity. The payout could be negative.

The usual definition of optimal play for Player 1 is the set of moves maximising their worst case payout, i.e. the ones which limit their losses if Player 2 selects the worst counter-move from Player 1's point of view.

In a zero sum game, or more generally in a constant sum game, this means the x which minimise $m(x)$, since the worst case response for Player 1 is the optimal response for Player 2.

In a non-constant sum game *this is not true*.

Player 1's strategy in a non-constant sum game should not assume optimal play from Player 2.

In no case does it make sense for Player 1 to be angry if Player 2 plays suboptimally.

Rock, Paper, Scissors

As a simple example, consider the game of rock, paper, scissors, or frog, slug, snake, or whatever name you know it under.
To make this interesting, and fair, both players should move simultaneously, but here we'll assume it's sequential.
Player 1 chooses from $X = \{0, 1, 2\}$.
Then Player 2 chooses from $Y = X$.
There are no restrictions, so $L = X \times Y$, i.e. $\Phi(x) = Y$ for all $x \in X$.
2 beats 1, which beats 0, which beats 2, so

$$\begin{aligned} f(0,1) = f(1,2) = f(2,0) &= 1, \\ f(1,0) = f(2,1) = f(0,2) &= -1, \\ f(0,0) = f(1,1) = f(2,2) &= 0. \end{aligned}$$

It's easy to see that $m(x) = 1$ for all x and $M(0) = \{1\}$, $M(1) = \{2\}$, and $M(2) = \{0\}$.
It doesn't matter what Player 1 does. Player 2 should respond in the obvious way.

Cake splitting

There is a cake of size 1. Player 1 splits the cake in two pieces. Player 2 chooses one of the pieces, leaving the other for Player 1.
$X = [0, 1]$, the set of possible sizes for piece 1.
$Y = \{1, 2\}$, the set of pieces Player 2 can choose.
$L = X \times Y$ or, equivalently, $\Phi(x) = Y$ for all $x \in X$.
$f(x, 1) = x$ while $f(x, 2) = 1 - x$.
$m(x) = \max_{y\in\Phi(y)} f(x, y) = \max\{x, 1 - x\} = \frac{1}{2} + \left|x - \frac{1}{2}\right|$.

$$M(x) = \begin{cases} \{2\} & \text{if } x < 1/2 \\ \{1,2\} & \text{if } x = 1/2 \\ \{1\} & \text{if } x > 1/2. \end{cases}$$

This is a constant-sum game. Player 1 is guaranteed a payout of size at least $1 - m(x) = \frac{1}{2} - \left|x - \frac{1}{2}\right|$, and so should choose $x = 1/2$.

Robot rock, paper, scissors

Suppose Player 1 and Player 2 both program robots to play for them by choosing probabilities for each move.
Player 2 chooses a program after Player 1 and so know's Player 1's robot's probabilities. Player 2's robot's probabilities can therefore depend on Player 1's robot's, but not on its actual move.

$$X = \{(p_0, p_1, p_2) \in \mathbf{R}^3 \colon p_0 + p_1 + p_2 = 1, p_0 \geq 0, p_1 \geq 0, p_2 \geq 0\}$$
$$Y = \{(q_0, q_1, q_2) \in \mathbf{R}^3 \colon q_0 + q_1 + q_2 = 1, q_0 \geq 0, q_1 \geq 0, q_2 \geq 0\}$$

$$f((p_0, p_1, p_2), (q_0, q_1, q_2)) = p_0 q_1 + p_1 q_2 + p_2 q_1 - q_0 p_1 - q_1 p_2 - q_2 p_1.$$

$$m(p_0, p_1, p_2) = \max\{p_2 - p_1, p_0 - p_2, p_1 - p_0\}.$$

$$M(p_0, p_1, p_2) = \{(q_0, q_1, q_2) \in Y : \forall j \colon p_{j-1} - p_{j+1} < m(p_0, p_1, p_2) \to q_j = 0\}.$$

The ultimatum game

Again there is a cake of size 1, but this time Player 1 splits the cake *and* chooses who gets which piece.
Player 2 can accept this arrangement or reject it, in which case no one gets cake.
$X = [0, 1]$, as before, but now x is the size of the slice Player 1 proposes to leave for Player 2.
$Y = \{0, 1\}$, where 0 means rejecting the deal and 1 means accepting it.
$L = X \times Y$ again.
$f(x, y) = xy$.
$m(x) = x$.
$M(0) = \{0, 1\}$ while $M(x) = \{1\}$ when $x > 0$
This is *not* a constant sum game.
In some sense $y = 1$ is the unique optimal Player 2 strategy for any positive x and so Player 1 can get a payout arbitrarily close to 1 by choosing x arbitrarily small, assuming Player 2 will play optimally.
I do not recommend this.

Berge's maximum theorem

The main point of Chapter 2 is the following theorem.

Theorem 2.23 (Berge's Maximum Theorem) *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, let $f\colon X \times Y \to \mathbf{R}$ be a continuous real-valued function on $X \times Y$, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi(\mathbf{x})$ is both non-empty and compact for all $\mathbf{x} \in X$ and that the correspondence $\Phi\colon X \rightrightarrows Y$ is both upper hemicontinuous and lower hemicontinuous. Let*

$$m(\mathbf{x}) = \sup\{f(\mathbf{x}, \mathbf{y})\colon \mathbf{y} \in \Phi(\mathbf{x})\}$$

for all $\mathbf{x} \in X$, and let

$$M(\mathbf{x}) = \{\mathbf{y} \in \Phi(\mathbf{x})\colon f(\mathbf{x}, \mathbf{y}) = m(\mathbf{x})\}$$

for all $\mathbf{x} \in X$. Then $m\colon X \to \mathbf{R}$ is continuous, $M(\mathbf{x})$ is a non-empty compact subset of Y for all $x \in X$, and the correspondence $M\colon X \rightrightarrows Y$ is upper hemicontinuous.

Comments on Berge's maximum theorem

The content of the theorem would be the same if we replaced sup by max.
Hopefully you noticed that the setup is like our game theory examples.
In addition to the proof in Part 1 of the notes there is a further proof in the appendix.
Both proofs use the lemmas, propositions, etc. we already discussed.
Usually when we have a theorem about a correspondence $\Phi\colon X \rightrightarrows Y$ we're interested in the special case where the correspondence comes from a function $\varphi\colon X \to Y$ via $\Phi(\mathbf{x}) = \{\varphi(\mathbf{x})\}$. Not here though.
In fact the most common case is $\Phi(\mathbf{x}) = Y$.
This is always upper and lower hemicontinuous. It's non-empty valued and compact-valued if and only if Y is non-empty and compact.