MAU34804

Lecture 5

2026-01-28

Can we strengthen Proposition 2.11?

Proposition 2.11 *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi(\mathbf{x})$ is closed in Y for every $\mathbf{x} \in X$. Suppose also that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. Then the graph $\mathrm{Graph}(\Phi)$ of $\Phi\colon X \rightrightarrows Y$ is closed in $X \times Y$.*

If we don't assume that Φ is closed-valued then the proposition is false.

$$\Phi(x) = (-1, 1)$$

is a counterexample.

If we assume lower hemicontinuity instead of upper hemicontinuity then the proposition is false.

$$\Phi(x) = \begin{cases} \{-1\} & \text{if } x < 0 \\ \varnothing & \text{if } x = 0 \\ \{1\} & \text{if } x > 0 \end{cases}$$

is a counterexample.

Can we strengthen Proposition 2.12?

Proposition 2.12 *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that the graph $\mathrm{Graph}(\Phi)$ of the correspondence Φ is closed in $X \times Y$. Suppose also that Y is a compact subset of $\mathbf{R}^m$. Then the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous.*

If we don't assume compactness then the proposition is false.

Example from notes:

$$f(x) = \begin{cases} 1/x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

has a closed graph but it is not continuous, so

$$F(x) = \begin{cases} \{1/x\} & \text{if } x > 0 \\ \{0\} & \text{if } x \leq 0 \end{cases}$$

is neither upper nor lower hemicontinuous.

Can we strengthen Proposition 2.12?

Proposition 2.12 *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that the graph $\mathrm{Graph}(\Phi)$ of the correspondence Φ is closed in $X \times Y$. Suppose also that Y is a compact subset of $\mathbf{R}^m$. Then the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous.*

What if we leave the hypotheses unchanged but add lower hemicontinuity to the conclusion?

Then

$$\Phi(x) = \begin{cases} \{-1\} & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ \{1\} & \text{if } x > 0 \end{cases}$$

is a counterexample.

Useful intermediate results

Some lemmas, propositions, etc. are intended for immediate use and can be forgotten after but some are worth remembering.

Proposition 2.6 *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and let G be a subset of $X \times Y$. Then G is closed in $X \times Y$ if and only if $(\lim_{j\to\infty} \mathbf{x}_j, \lim_{j\to\infty} \mathbf{y}_j) \in G$ for all convergent infinite sequences $\mathbf{x}_1, \mathbf{x}_2, \ldots$ in X and for all convergent infinite sequences $\mathbf{y}_1, \mathbf{y}_2, \ldots$ in Y with the property that $(\mathbf{x}_j, \mathbf{y}_j) \in G$ for all positive integers j.*

In general I don't recommend sequential criteria for closedness, compactness, continuity, etc. but it's sometimes useful and perhaps intuitive.

Proposition 2.9 *Let X be a subset of n-dimensional Euclidean space $\mathbf{R}^n$, let V be a subset of X that is open in X, and let K be a compact subset of $\mathbf{R}^n$ satisfying $K \subset V$. Then there exists some positive real number ϵ with the property that $B_X(K, \epsilon) \subset V$, where $B_X(K, \epsilon)$ denotes the subset of X consisting of those points of X that lie within a distance less than ϵ of some point of K.*

An alternate way to say this is that if a closed set and compact set are disjoint then the distance between them is positive.

More facts about compact-valued correspondences, (2.15)

Three useful facts are proved in Section 2.3 of the notes.

Lemma 2.15 *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is compact-valued and upper hemicontinuous. Let K be a compact subset of X. Then $\Phi(K)$ is a compact subset of Y.*

$\Phi(K)$ means $\cup_{\mathbf{x}\in K}\Phi(x)$. Really one should write $\Phi_{-}(K)$ instead, but no one does.

This is the analogue of the fact that the image of a compact set under a continuous function is compact.

It's false if we substitute lower hemicontinuity for upper hemicontinuity.

A counterexample is $K = [-1, 1]$ and $G\colon \mathbf{R} \rightrightarrows \mathbf{R}$ defined by

$$G(x) = \begin{cases} \varnothing & \text{if } \ = 0 \\ \{1/x\} & \text{if } x \neq 0. \end{cases}$$

G is lower hemicontinuous at 0 because any correspondence is locally hemicontinuous at any point where it takes the value $\varnothing$ and it's lower semicontinuous everywhere else because it corresponds, locally, to a continuous function.

Of course $\Phi(K) = (-\infty, -1] \cup [1, \infty)$ is not compact.

More facts about compact-valued correspondences, (2.16)

Proposition 2.16 *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a compact-valued correspondence from X to Y. Let $\mathbf{p}$ be a point of X for which $\Phi(\mathbf{p})$ is non-empty. Then the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous at $\mathbf{p}$ if and only if, given any positive real number ϵ, there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset B_Y(\Phi(\mathbf{p}), \epsilon)$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$, where $B_Y(\Phi(\mathbf{p}), \epsilon)$ denotes the subset of Y consisting of all points of Y that lie within a distance ϵ of some point of $\Phi(\mathbf{p})$.*

Three criteria for continuity of continuous functions are the δ-ϵ definition, the fact that preimages of open sets are open, and the sequential criterion.

This proposition corresponds to the δ-ϵ criterion.

Lemma 2.1, which says that Φ is continuous if and only if Φ^+ takes open sets to open sets, corresponds to the open set criterion.

What about the actual definition given in the notes for upper hemicontinuity? It corresponds to the hybrid criterion that $\varphi\colon X \to Y$ is continuous at $p \in X$ if for all open sets V containing $\varphi(p)$ there is a $\delta > 0$ such that if $|p - x| < \delta$ then $\phi(x) \in V$.

More facts about compact-valued correspondences, (2.17)

Proposition 2.17 *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Then the correspondence is both compact-valued and upper hemicontinuous at a point $\mathbf{p} \in X$ if and only if, given any infinite sequences $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ and $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ in X and Y respectively, where $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim_{j\to+\infty} \mathbf{x}_j = \mathbf{p}$, there exists a subsequence of $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ which converges to a point of $\Phi(\mathbf{p})$.*

This, combined with the following theorem from Section 2.4, is about as close as you'll get to an analogue of the sequential criterion for continuity of functions.

Proposition 2.19 *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is lower hemicontinuous at a point $\mathbf{p}$ of X if and only if given any infinite sequence $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3, \ldots$ in X for which $\lim_{j\to+\infty} \mathbf{x}_j = \mathbf{p}$ and given any point $\mathbf{q}$ of $\Phi(\mathbf{p})$, there exists an infinite sequence $\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots$ of points of Y such that $\mathbf{y}_j \in \Phi(\mathbf{x}_j)$ for all positive integers j and $\lim_{j\to+\infty} \mathbf{y}_j = \mathbf{q}$.*

Proposition 2.19 is only ever used to give an alternate proof of the Berge Maximum Theorem in the Appendix so Section 2.4 can be safely ignored.

An open mapping theorem

The main point of Section 2.3 is the following theorem.

Proposition 2.18 *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper hemicontinuous and compact-valued. Let U be an open set in $X \times Y$. Then*

$$\{\mathbf{x} \in X\colon (\mathbf{x}, \mathbf{y}) \in U \text{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X.

In addition to the proof in Part I of the notes there are two further proofs in the Appendix.

The analogue for continuous functions is the statement that for any continuous function $\varphi\colon X \to Y$ and and open U in $X \times Y$ the set $\{\mathbf{x}\colon (\mathbf{x}, \varphi(\mathbf{x})) \in U\}$ is open.

A simple proof is that for any continuous functions $\theta\colon X \to Z$ and $\varphi\colon X \to Y$ the function $\psi\colon X \to Z \times Y$ defined by $\psi(\mathbf{x}) = (\theta(\mathbf{x}), \varphi(\mathbf{x}))$ is continuous and

$$\{\mathbf{x}\colon (\mathbf{x}, \varphi(\mathbf{x})) \in U\} = \psi^*(U),$$

where ψ is as above with $Z = X$ and $\theta\colon X \to X$ just being the identity function.

Yet another proof of 2.15

Proposition 2.18 *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y that is both upper hemicontinuous and compact-valued. Let U be an open set in $X \times Y$. Then*

$$\{\mathbf{x} \in X \colon (\mathbf{x}, \mathbf{y}) \in U \textit{ for all } \mathbf{y} \in \Phi(\mathbf{x})\}$$

is open in X.

One can prove this by the method used on the previous slide for continuous functions. First one proves a lemma the if K and L are compact subsets of Z and Y and W is an open subset of $Z \times Y$ such that $K \times L \subset W$ then there are open subsets U and V of X and Y such that $K \subset U$ and $L \subset V$.

Next one proves that if $\Theta\colon X \rightrightarrows Z$ and $\Phi\colon X \rightrightarrows Y$ are compact-valued upper hemicontinuous then so is $\Psi\colon X \rightrightarrows Z \times Y$, defined by $\Psi(\mathbf{x}) = \Theta(\mathbf{x}) \times \Phi(\mathbf{x})$.

Finally one notes that $\{\mathbf{x} \colon \forall y \in \Phi(\mathbf{x}) \colon (\mathbf{x}, \mathbf{x})) \in U\} = \Psi^+(U)$, with $Z = X$ and $\Theta(\mathbf{x}) = \{\mathbf{x}\}$.

Intersections of Correspondences

The following proposition is proved in Section 2.5.

Proposition 2.20 *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, let $\Phi\colon X \rightrightarrows Y$ and $\Psi\colon X \rightrightarrows Y$ be correspondences from X to Y, where the correspondence $\Phi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous and the correspondence $\Psi\colon X \rightrightarrows Y$ has closed graph. Let $\Phi \cap \Psi\colon X \rightrightarrows Y$ be the correspondence defined such that*

$$(\Phi \cap \Psi)(\mathbf{x}) = \Phi(\mathbf{x}) \cap \Psi(\mathbf{x})$$

for all $\mathbf{x} \in X$. Then the correspondence $\Phi \cap \Psi\colon X \rightrightarrows Y$ is compact-valued and upper hemicontinuous.

There is no non-trivial analogue for continuous functions.