MAU34804

Lecture 4

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Φ^+ and composition

For functions we have $(g \circ f)^* = f^* \circ g^*$. What about correspondences? Do we have $(G \circ F)^+ = F^+ \circ G^+$?

$\mathbf{x} \in F^+(U)$ means $F(\mathbf{x}) \subset U$, which means that for all $\mathbf{y}$, if $\mathbf{y} \in F(\mathbf{x})$ then $\mathbf{y} \in U$.

In logical notation,

$$\forall \mathbf{y} \colon \mathbf{y} \in F(\mathbf{x}) \to \mathbf{y} \in U.$$

Similarly, $\mathbf{y} \in G^+(V)$ is, in logical notation,

$$\forall \mathbf{z} \colon \mathbf{z} \in G(\mathbf{y}) \to \mathbf{z} \in V.$$

Taking $U = G^+(V)$ we find that $\mathbf{x} \in F^+(G^+(V))$ means

$$\forall \mathbf{y} \colon \mathbf{y} \in F(\mathbf{x}) \to \mathbf{y} \in G^+(V),$$

or, in view of what we said earlier about $G^+(V)$,

$$\forall \mathbf{y} \colon \mathbf{y} \in F(\mathbf{x}) \to \left(\forall \mathbf{z} \colon \mathbf{z} \in G(\mathbf{y}) \to \mathbf{z} \in V\right).$$

Φ^+ and composition, continued

Replacing U with V and F with $G \circ F$ in our first equivalence we get that $\mathbf{x} \in (G \circ F)^+(V)$ is equivalent to

$$\forall \mathbf{z} \colon \mathbf{z} \in (G \circ F)(\mathbf{x}) \to \mathbf{z} \in V.$$

The definition of composition of correspondences was that $\mathbf{z} \in (G \circ F)(\mathbf{x})$ means

$$\exists \mathbf{y} \colon \mathbf{y} \in F(\mathbf{x}) \wedge \mathbf{z} \in G(\mathbf{y})$$

Substituting, $\mathbf{x} \in (G \circ F)^+(V)$ is equivalent to

$$\forall \mathbf{z} \colon \ (\exists \mathbf{y} \colon \mathbf{y} \in F(\mathbf{x}) \wedge \mathbf{z} \in G(\mathbf{y})) \to \mathbf{z} \in V$$

Is this equivalent to

$$\forall \mathbf{y} \colon \mathbf{y} \in F(\mathbf{x}) \to (\forall \mathbf{z} \colon \mathbf{z} \in G(\mathbf{y}) \to \mathbf{z} \in V)?$$

Φ^+ and composition, conclusion

We've seen that $\mathbf{x} \in F^+(G^+(V)$ means

$$\forall \mathbf{y} \colon p(\mathbf{x}, \mathbf{y}) \to (\forall \mathbf{z} \colon q(\mathbf{y}, \mathbf{z}) \to r(\mathbf{z}))\,,$$

while $\mathbf{x} \in (G \circ F)^+(V)$ is equivalent to

$$\forall \mathbf{z} \colon\ (\exists \mathbf{y} \colon p(\mathbf{x}, \mathbf{y}) \wedge q(\mathbf{y}, \mathbf{z})) \to r(\mathbf{z}),$$

where

$$p(\mathbf{x}, \mathbf{y}) = \mathbf{y} \in F(\mathbf{x}), \quad q(\mathbf{y}, \mathbf{z}) = \mathbf{z} \in G(\mathbf{y}), \quad r(\mathbf{z}) = \mathbf{z} \in V.$$

Are these two statements equivalent?
Yes, no matter what statements are represented by $p(\mathbf{x}, \mathbf{y})$, $q(\mathbf{y}, \mathbf{z})$, and $r(\mathbf{z})$.
This is an exercise in first order logic. I'll do it on the next slide.
It follows that $(G \circ F)^+ = F^+ \circ G^+$.
From this it follows that the composition of upper hemicontinuous correspondences is upper hemicontinuous.

First order logic

The following statements are all equivalent

$$
\begin{aligned}
&\forall \mathbf{z}\colon\ (\exists \mathbf{y}\colon p(\mathbf{x},\mathbf{y}) \land q(\mathbf{y},\mathbf{z})) \rightarrow r(\mathbf{z})\\
&\forall \mathbf{z}\colon \neg(\exists \mathbf{y}\colon p(\mathbf{x},\mathbf{y}) \land q(\mathbf{y},\mathbf{z})) \lor r(\mathbf{z})\\
&\forall \mathbf{z}\colon \forall \mathbf{y}\colon \neg(p(\mathbf{x},\mathbf{y}) \land q(\mathbf{y},\mathbf{z})) \lor r(\mathbf{z})\\
&\forall \mathbf{z}\colon \forall \mathbf{y}\colon \neg p(\mathbf{x},\mathbf{y}) \lor \neg q(\mathbf{y},\mathbf{z}) \lor r(\mathbf{z})\\
&\forall \mathbf{z}\colon \forall \mathbf{y}\colon \neg p(\mathbf{x},\mathbf{y}) \lor q(\mathbf{y},\mathbf{z}) \rightarrow r(\mathbf{z})\\
&\forall \mathbf{z}\colon \forall \mathbf{y}\colon p(\mathbf{x},\mathbf{y}) \rightarrow q(\mathbf{y},\mathbf{z}) \rightarrow r(\mathbf{z})\\
&\forall \mathbf{y}\colon \forall \mathbf{z}\colon p(\mathbf{x},\mathbf{y}) \rightarrow q(\mathbf{y},\mathbf{z}) \rightarrow r(\mathbf{z})\\
&\forall \mathbf{y}\colon p(\mathbf{x},\mathbf{y}) \rightarrow (\forall \mathbf{z}\colon q(\mathbf{y},\mathbf{z}) \rightarrow r(\mathbf{z}))
\end{aligned}
$$

Φ^- and composition

We can do the same thing with Φ^-. It turns out that $\mathbf{x} \notin F^-(G^-(V)$ means

$$\forall \mathbf{y} \colon p(\mathbf{x}, \mathbf{y}) \to (\forall \mathbf{z} \colon q(\mathbf{y}, \mathbf{z}) \to r(\mathbf{z}))\,,$$

while $\mathbf{x} \notin (G \circ F)^+(V)$ is equivalent to

$$\forall \mathbf{z} \colon (\exists \mathbf{y} \colon p(\mathbf{x}, \mathbf{y}) \wedge q(\mathbf{y}, \mathbf{z})) \to r(\mathbf{z}),$$

where

$$p(\mathbf{x}, \mathbf{y}) = \mathbf{y} \notin F(\mathbf{x}), \quad q(\mathbf{y}, \mathbf{z}) = \mathbf{z} \notin G(\mathbf{y}), \quad r(\mathbf{z}) = \mathbf{z} \notin V.$$

From the same bit of first order logic it follows that $(G \circ F)^- = F^- \circ G^-$ and thus that the composition of lower hemicontinuous correspondences is lower hemicontinuous.

Graphs of functions

Lemma 2.7 *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. Suppose that $\varphi\colon X \to Y$ is continuous. Then the graph $\mathrm{Graph}(\varphi)$ of the function φ is closed in $X \times Y$.*

There are multiple ways to prove this. For example, $\mathrm{Graph}(\varphi) = \psi^*(\{\mathbf{0}\})$, where $\psi(\mathbf{x}, \mathbf{y}) = \mathbf{y} - \varphi(\mathbf{x})$.

ψ is continuous and $\{\mathbf{0}\}$ is closed. The preimage of a closed set under a continuous function is closed.

Alternatively, suppose $(\mathbf{x}, \mathbf{y})$ belongs to the complement of $\mathrm{Graph}(\varphi)$ in $X \times Y$, i.e that $\mathbf{y} \neq \varphi(\mathbf{x})$.

Y is Hausdorff, so there are disjoint open neighbourhoods U and V of $\varphi(\mathbf{x})$ and $\mathbf{y}$ in Y.

Then $\varphi^*(U) \times V$ is an open neighourhood of $(\mathbf{x}, \mathbf{y})$ which is contained in the complement of $\mathrm{Graph}(\varphi)$ in $X \times Y$.

Since every point in the complement of the graph has such an open neighbourhood the complement is open, which means the graph is closed.

X and Y don't need to be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$; Y just needs to be Hausdorff.

Graphs of functions, continued

Corollary 2.13 *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and let $\varphi\colon X \to Y$ be a function from X to Y. Suppose that the graph $\mathrm{Graph}(\varphi)$ of the function φ is closed in $X \times Y$. Suppose also that Y is a compact subset of $\mathbf{R}^m$. Then the function $\varphi\colon X \to Y$ is continuous.*

In the notes this is a corollary to Propositions 2.11 and 2.12 on correspondences but we can proved it directly.

Suppose V is an open subset of Y and $\mathbf{x} \in \varphi^*(V)$, so $\varphi(\mathbf{x}) \in V$.

Let K be the complement of V in Y and let O be the complement of $\mathrm{Graph}(\varphi)$ in $X \times Y$. Then O is open and K is closed. In fact K is compact, since it is a closed subset of a compact set.

For each $\mathbf{y} \in K$ we have $(\mathbf{x}, \mathbf{y}) \in O$ since $\mathbf{y} \neq \varphi(\mathbf{x})$. Now O is open so there are open neighbourhoods U and W of $\mathbf{x}$ and $\mathbf{y}$ in X and Y such that $U \times W \subset O$.

Each $\mathbf{y} \in K$ belongs to some such W so they form an open cover of K.

K is compact so we can find a finite subcover. In other words, there are neighbourhoods $U_1, \ldots, U_l$ of $\mathbf{x}$ in X and open sets $W_1, \ldots, W_l$ in Y such that for all j we have $U_j \times W_j \subset O$. Let $U = \bigcap_{j=1}^{l} U_j$ and $W = \bigcup_{j=1}^{l} W_j$.

Proof of Corollary 2.13, continued

Let $U = \bigcap_{j=1}^{l} U_j$ and $W = \bigcup_{j=1}^{l} W_j$.
U is an open neighbourhood of $\mathbf{x}$ and $K \subset W$.
Suppose $\mathbf{q} = \varphi(\mathbf{p})$, i.e. that $(\mathbf{p}, \mathbf{q}) \in \mathrm{Graph}(\varphi)$, where $\mathbf{p} \in U$.
Then $\mathbf{p} \in U_j$ for all j,
Now O is the complement of $\mathrm{Graph}(\varphi)$ so $(\mathbf{p}, \mathbf{q}) \notin O$.
$U_j \times W_j \subset O$ so $(\mathbf{p}, \mathbf{q}) \notin U_j \times W_j$, but $\mathbf{p} \in U_j$ so $\mathbf{q} \notin W_j$.
Now $K \subset W$ and $W = \cup_{j=1}^{l} W_j$ so $\mathbf{q} \notin K$. But K is the complement of V so $\mathbf{q} \in V$.
In other words, $\varphi(\mathbf{p}) \in V$, so $\mathbf{p} \in \varphi^*(V)$.
So if $\mathbf{p} \in U$ then $\mathbf{p} \in \varphi^*(V)$, which means $U \subset \varphi^*(V)$.
So every point $\mathbf{x} \in \varphi^*(V)$ has a neighbourhood U contained in $\varphi^*(V)$ and therefore $\varphi^*(V)$ is open.
This is true for all open V so φ is continuous. This completes the proof of Corollary 2.13.
X and Y don't need to be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$; Y just needs to be compact.

Graphs of correspondences

A correspondence $\Phi\colon X \rightrightarrows Y$ is a function from X to $\mathcal{P}(Y)$ so its graph should be the set of points $(\mathbf{x}, \Phi(\mathbf{y}))$ in $X \times \mathcal{P}(Y)$.
Unfortunately this is not what people mean by the term. Instead they mean

$$\mathrm{Graph}(F) = \{(\mathbf{x}, \mathbf{y}) \in X \times Y : \mathbf{y} \in F(\mathbf{x})\}.$$

As discussed previously, every correspondence from X to Y corresponds to a relation between X and Y, i.e. to a subset of the product $X \times Y$. The graph, as defined above, *is* that relation, just as the usual graph of a function *is* the function.

Hemicontinuity and closed graphs

Proposition 2.11 *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that $\Phi(\mathbf{x})$ is closed in Y for every $x \in X$. Suppose also that $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous. Then the graph $\mathrm{Graph}(\Phi)$ of $\Phi\colon X \rightrightarrows Y$ is closed in $X \times Y$.*

The only property of Y we need is that it is regular, i.e. that any point and any closed set not containing it can be separated by closed sets. All subsets of Euclidean space are regular.

Suppose $(\mathbf{x}, \mathbf{y})$ belongs to the complement of $\mathrm{Graph}(\Phi)$ in $X \times Y$, i.e that $\mathbf{y} \notin \Phi(\mathbf{x})$.

Y is regular and $\Phi(\mathbf{x})$ is closed so there are disjoint open sets U and V in Y with $\Phi(\mathbf{x}) \subset U$ and $\mathbf{y} \in V$.

Then $\Phi^+(U) \times V$ is an open neighbourhood of $(\mathbf{x}, \mathbf{y})$.

If $(\mathbf{p}, \mathbf{q}) \in \Phi^+(U) \times V$ then $\mathbf{p} \in \Phi^+(U)$ so $\Phi(\mathbf{p}) \subset U$.

$\mathbf{q} \in V$ and U and V are disjoint so $\mathbf{q} \notin \Phi(\mathbf{p})$.

Thus $\Phi^+(U) \times V$ is contained in the complement of $\mathrm{Graph}(\Phi)$ in $X \times Y$.

Since every point in the complement of the graph has such an open neighbourhood the complement is open, which means the graph is closed.

Hemicontinuity and closed graphs, continued

Proposition 2.12 *Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively, and let $\Phi\colon X \rightrightarrows Y$ be a correspondence from X to Y. Suppose that the graph $\mathrm{Graph}(\Phi)$ of the correspondence Φ is closed in $X \times Y$. Suppose also that Y is a compact subset of $\mathbf{R}^m$. Then the correspondence $\Phi\colon X \rightrightarrows Y$ is upper hemicontinuous.*

Suppose V is an open subset of Y and $\mathbf{x} \in \Phi^+(V)$, so $\Phi(\mathbf{x}) \subset V$.

Let K be the complement of V in Y and let O be the complement of $\mathrm{Graph}(\Phi)$ in $X \times Y$. Then O is open and K is closed, and in fact compact.

For each $\mathbf{y} \in K$ we have $(\mathbf{x}, \mathbf{y}) \in O$ since $\mathbf{y} \notin \Phi(\mathbf{x})$. Now O is open so there are open neighbourhoods U and W of $\mathbf{x}$ and $\mathbf{y}$ in X and Y such that $U \times W \subset O$.

Each $\mathbf{y} \in K$ belongs to some such W so they form an open cover of K.

K is compact so we can find a finite subcover. In other words, there are neighbourhoods $U_1, \ldots, U_l$ of $\mathbf{x}$ in X and open sets $W_1, \ldots, W_l$ in Y such that for all j we have $U_j \times W_j \subset O$. Let $U = \bigcap_{j=1}^{l} U_j$ and $W = \bigcup_{j=1}^{l} W_j$.

U is an open neighbourhood of $\mathbf{x}$ and $K \subset W$.

Suppose $\mathbf{q} \in \Phi(\mathbf{p})$, i.e. that $(\mathbf{p}, \mathbf{q}) \in \mathrm{Graph}(\Phi)$, where $\mathbf{p} \in U$. Then $\mathbf{p} \in U_j$ for all j.

Proof of Proposition 2.12, continued

O is the complement of $\mathrm{Graph}(\Phi)$ so $(\mathbf{p}, \mathbf{q}) \notin O$.
$U_j \times W_j \subset O$ so $(\mathbf{p}, \mathbf{q}) \notin U_j \times W_j$, but $\mathbf{p} \in U_j$ so $\mathbf{q} \notin W_j$.
Now $K \subset W$ and $W = \cup_{j=1}^{l} W_j$ so $\mathbf{q} \notin K$. But K is the complement of V so $\mathbf{q} \in V$.
This holds for all $\mathbf{q} \in \Phi(\mathbf{p})$, so $\Phi(\mathbf{p}) \subset V$. In other words, $\mathbf{p} \in \Phi^{+}(V)$.
So if $\mathbf{p} \in U$ then $\mathbf{p} \in \Phi^{+}(V)$, which means $U \subset \Phi^{+}(V)$.
So every point $\mathbf{x} \in \Phi^{+}(V)$ has a neighbourhood U contained in $\Phi^{+}(V)$ and therefore $\varphi^{+}(V)$ is open.
This holds for all open V so Φ is continuous, completing the proof of Proposition 2.12.