MAU34804

Lecture 3

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Using locality

For any subset S of $[-1, 1]$, consider the correspondence

$$\Phi(x) = \begin{cases} \{-1\} & \text{if } x < 0 \\ S & \text{if } x = 0 \\ \{1\} & \text{if } x > 0. \end{cases}$$

This is, of course a generalisation of the previous two examples.

Under what conditions on S is Φ upper or lower hemicontinuous? Can it be both?

The restriction of Φ to $[-1, 0) \cup (0, 1]$ is both upper and lower hemicontinuous, by the restriction theorem, and the fact that $\varphi(x) = x/|x|)$ is continuous on $[-1, 0) \cup (0, 1]$.

So the only thing we need to check is hemicontinuity at 0.

Upper hemicontinuity at 0 is the statement that if $S \subset V$ then $\Phi(x) \subset V$ for all sufficiently small x. Since $\Phi(x)$ can be either $\{-1\}$ or $\{1\}$ we need $\{-1, 1\} \subset V$.

So what we need is that every open V containing S contains $\{-1, 1\}$. This happens if and only if $\{-1, 1\} \subset S$.

Using locality, continued

That answers the question about upper hemicontinuity. What about lower hemicontinuity?

For lower hemicontinuity at 0 we need that if $S \cap V \neq \varnothing$ then $\Phi(x) \cap V \neq \varnothing$ for all sufficiently small x. Again $\Phi(x)$ can be either $\{-1\}$ or $\{1\}$.

Consider $V_+ = (-1, 1]$. Note that this is open in $[-1, 1]$!

V_+ has empty intersection with $\Phi(x)$ for all small negative x so S must also have have empty intersection with V_+.

Similarly, $V_- = [-1, 1)$ has empty intersection with $\Phi(x)$ for all small positive x so S must also have have empty intersection with V_-.

The only subset which has empty intersection with both is $S = \varnothing$, so the only lower hemicontinuous example is the one we already saw.

$\{-1, 1\}$ is not a subset of $\varnothing$ so no S makes Φ both upper and lower hemicontinuous.

Yet another example

I mentioned before that $\leq$ is a relation on $\mathbf{R}$. The corresponding correspondence $\Phi\colon \mathbf{R} \rightrightarrows \mathbf{R}$ is $\Phi(x) = [x, \infty)$.

I claim that Φ is both upper and lower hemicontinuous.

To show that Φ is upper hemicontinuous at p we need to show that if $\Phi(p) = [p, \infty) \subset V$ for some open V then there is a $\delta > 0$ such that $\Phi(x) = [x, \infty) \subset V$ for all x such that $|p - x| < \delta$.

$p \in V$ is open so there is an r such that $B_{\mathbf{R}}(p, r) \subset V$. Let $\delta = r$. If $|p - x| < \delta$ then

$$[x, \infty) \subset [p, \infty) \cup B_{\mathbf{R}}(p, r) \subset V.$$

To show that Φ is lower hemicontinuous at p we need to show that if $[p, \infty) \cap V \neq \varnothing$ then there is some $\delta > 0$ such that if $|p - x| < \delta$ then $[x, \infty) \cap V \neq \varnothing$.

$[p, \infty) \cap V \neq \varnothing$ just means there is a $y \in V$ such that $p \leq y$. V is open, so there is an $r > 0$ such that $B_{\mathbf{R}}(y, r) \subset V$.

In particular $z = y + r/2 \in V$. Let $\delta = r/2$. If $|p - x| < \delta$ then $z = y + \delta \geq p + \delta \geq x$ so $z \in [x, \infty) \cap V$.

Φ^+ and Φ^-

For any function $\varphi\colon X \to Y$ we have a function $\varphi_*\colon \mathcal{P}(X) \to \mathcal{P}(Y)$ and a function $\varphi^*\colon \mathcal{P}(Y) \to \mathcal{P}(X)$ defined by

$$\varphi_*(U) = \{y \in Y\colon \exists x \in X\colon \varphi(x) = y\}, \quad \varphi^*(V) = \{x \in X\colon \varphi(x) \in V\}.$$

People write $\varphi(U)$ for $\varphi_*(U)$ and write $\varphi^{-1}(V)$ for $\varphi^*(V)$, but this is dangerous. Recall that φ^* is better behaved than φ_*, e.g. that $\varphi^*(V \cap W) = \varphi^*(V) \cap \varphi^*(W)$, but we only have $\varphi_*(V \cap W) \subset \varphi_*(V) \cap \varphi_*(W)$.
Just as continuity of functions splits into two notions for correspondences, φ^* has two different counterparts for a correspondence $\Phi\colon X \rightrightarrows Y$:

$$\Phi^+(V) = \{x \in X\colon \Phi(x) \subset V\}, \quad \Phi^-(V) = \{x \in X\colon \Phi(x) \cap V \neq \varnothing\}.$$

φ is continuous if φ^* takes open subsets of Y to open subsets of X.
Φ is upper hemicontinuous if Φ^+ takes open subsets of Y to open subsets of X.
Φ is lower hemicontinuous if Φ^- takes open subsets of Y to open subsets of X.

Properties of Φ^+

If $\varphi\colon X \to Y$ is a function then φ^* has the useful properties

$$\varphi^*(U \cup V) = \varphi^*(U) \cup \varphi^*(V), \quad \varphi^*(U \cap V) = \varphi^*(U) \cap \varphi^*(V)$$

What about Φ^+ and Φ^-, when $\Phi\colon X \rightrightarrows Y$ is a correspondence?
Suppose $x \in \Phi^+(U) \cap \Phi^+(V)$. Then $x \in \Phi^+(U)$ and $x \in \Phi^+(V)$.
In other words, $\Phi(x) \subset U$ and $\Phi(x) \subset V$. But then $\Phi(x) \subset U \cap V$, so $x \in \Phi^+(U \cap V)$.
If $x \in \Phi^+(U) \cap \Phi^+(V)$ then $\Phi(x) \subset U \cap V$, so $x \in \Phi^+(U \cap V)$.
So if $x \in \Phi^+(U) \cap \Phi^+(V)$ then $x \in \Phi^+(U \cap V)$ or, in other words,
$\Phi^+(U) \cap \Phi^+(V) \subset \Phi^+(U \cap V)$.
Suppose $x \in \Phi^+(U) \cup \Phi^+(V)$. Then $x \in \Phi^+(U)$ or $x \in \Phi^+(V)$.
In other words, $\Phi(x) \subset U$ or $\Phi(x) \subset V$. But then $\Phi(x) \subset U \cup V$, so $x \in \Phi^+(U \cup V)$.
If $x \in \Phi^+(U) \cup \Phi^+(V)$ then $\Phi(x) \subset U \cup V$, so $x \in \Phi^+(U \cup V)$.
So if $x \in \Phi^+(U) \cup \Phi^+(V)$ then $x \in \Phi^+(U \cup V)$ or, in other words,
$\Phi^+(U) \cup \Phi^+(V) \subset \Phi^+(U \cup V)$.
These arguments are almost identical, but the first one is reversible, so in fact
$\Phi^+(U) \cap \Phi^+(V) = \Phi^+(U \cap V)$, but the second is not reversible.

Properties of Φ^-

Suppose that $x \in \Phi^-(U \cup V)$. Then $\Phi(x)$ has non-empty interesection with $U \cup V$.
$\Phi(x)$ must then have non-empty intersection with U or with V.
In other words, $x \in \Phi^-(U)$ or $x \in \Phi^-(V)$.
So $x \in \Phi^-(U) \cup \Phi^-(V)$.
If $x \in \Phi^-(U \cup V)$ then $x \in \Phi^-(U) \cup \Phi^-(V)$ or, in other words
$\Phi^-(U \cup V) \subset \Phi^-(U) \cup \Phi^-(V)$.
Suppose that $x \in \Phi^-(U \cap V)$. Then $\Phi(x)$ has non-empty interesection with $U \cap V$.
$\Phi(x)$ must then have non-empty intersection with U and with V.
In other words, $x \in \Phi^-(U)$ and $x \in \Phi^-(V)$.
So $x \in \Phi^-(U) \cap \Phi^-(V)$.
If $x \in \Phi^-(U \cap V)$ then $x \in \Phi^-(U) \cap \Phi^-(V)$ or, in other words
$\Phi^-(U \cap V) \subset \Phi^-(U) \cap \Phi^-(V)$.
These arguments too are almost identical, but the first one is reversible, and the second isn't but the second is not reversible, so we get $\Phi^-(U \cup V) = \Phi^-(U) \cup \Phi^-(V)$, but not the corresponding statement for intersections.
Neither Φ^+, nor Φ^- is a perfect analogue of φ^*, but they fail in different ways.

Composition of correspondences

If $f\colon X \to Y$ and $g\colon Y \to Z$ are functions then their composition is defined by $(g \circ f)(x) = g(f(x))$.

Functions are a special kind of relation. $y = f(x)$ means $(x, y) \in f$, $z = g(y)$ means $(y, z) \in g$, and $z = (g \circ f)(x)$ means $(x, z) \in g \circ f$.

So $(g \circ f)(x) = g(f(x))$ is equivalent to the statement that $(x, z) \in g \circ f$ if and only if there is a $y \in Y$ such that $(x, y) \in f$ and $(y, z) \in g$.

This condition makes sense for all types of relations, not just functions, and is the standard definition of composition of relations.

The connection between relations and correspondences is that $F(x) = \{y \in Y \colon (x, y) \in f\}$ and $(x, y) \in f$ if and only if $y \in F(x)$.

Translating, we get the definition of composition of correspondences:

$$(G \circ F)(x) = \{z \in Z \colon \exists y \in F(x) \colon z \in G(y)\}.$$

$G \circ F\colon X \rightrightarrows Z$ is a function from X to $\mathcal{P}(Z)$, and so isn't the composition of G, a function from Y to $\mathcal{P}(Z)$, and F, a function from X to $\mathcal{P}(Y)$, but there's no composition of G and F so saying composition and writing $\circ$ rarely causes confusion.