MAU34804

Lecture 2

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Why I hate correspondences, but will talk about them all semester

Functions are a very important type of relation. To every relation there corresponds a correspondence, as we just saw. Correspondences are a type of function.

Question: If I take a function, view it as a relation, convert it to a correspondence, and view that as function, do I get back the same function I started with?

Answer: No! If I started with a function from X to Y I get a function from X to $\mathcal{P}(Y)$. These functions are related by

$$F(x) = \{f(x)\}.$$

Note that y and $\{y\}$ are generally not the same thing.

You can sort of get away with ignoring the distinction as long as y is not a set, i.e. if you don't need to talk about sets of sets, but that's exactly what $\mathcal{P}(X)$ is!

Life would be simpler and less confusing without correspondences but that's the language people in the field use, so we have to adapt to it.

Upper and lower hemicontinuity of correspondences

Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is said to be *upper hemicontinuous* at a point $\mathbf{p}$ of X if, given any set V in Y that is open in Y and satisfies $\Phi(\mathbf{p}) \subset V$, there exists some positive real number δ such that $\Phi(\mathbf{x}) \subset V$ for all $\mathbf{x} \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. The correspondence Φ is upper hemicontinuous on X if it is upper hemicontinuous at each point of X.

Let X and Y be subsets of $\mathbf{R}^n$ and $\mathbf{R}^m$ respectively. A correspondence $\Phi\colon X \rightrightarrows Y$ is said to be *lower hemicontinuous* at a point $\mathbf{p}$ of X if, given any set V in Y that is V open in Y and satisfies $\Phi(\mathbf{p}) \cap V \neq \varnothing$, there exists some positive real number δ such that $\Phi(\mathbf{x}) \cap V \neq \varnothing$ for all $x \in X$ satisfying $|\mathbf{x} - \mathbf{p}| < \delta$. The correspondence Φ is lower hemicontinuous on X if it is lower hemicontinuous at each point of X.

These definitions are even more similar than they look.

$\Phi(\mathbf{x}) \subset V$ means $\mathbf{y} \in V$ for *all* $\mathbf{y} \in (mathbfx)$. $\Phi(\mathbf{x}) \cap V \neq \varnothing$ means $\mathbf{y} \in V$ for *some* $\mathbf{y} \in (mathbfx)$.

Note that if $\Phi(\mathbf{x}) = \{\mathbf{y}\}$ then these conditions are equivalent.

This is precisely what happens when the correspondence $\Phi\colon X \rightrightarrows Y$ corresponds to a function $\varphi\colon X \to Y$.

Comments on upper and lower hemicontinuity

For functions from a subset $\mathbf{R}^n$ to a subset of $\mathbf{R}$ we also have the concept of upper and lower semicontinuity. This is unrelated to upper and lower hemicontinuity.

We've seen that upper and lower hemicontinuity are equivalent for correspondences Φ coming from functions φ. This is not true in general.

Let $X = Y = Z = [-1, 1] \subset \mathbf{R}$ and say that $\Phi(x)$ is the set of maximisers of the linear function xz for $z \in Z$, i.e.

$$\Phi(x) = \begin{cases} \{-1\} & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ \{1\} & \text{if } x > 0 \end{cases}$$

I claim Φ is upper hemicontinuous but not lower hemicontinuous.

To show that Φ is upper hemicontinuous I need to find, for each $p \in X$ and any open superset of $\Phi(p)$ in Y, a $\delta > 0$ such that $|p - x| < \delta$ implies $\Phi(x) \subset V$.

To show that Φ is not lower hemicontinuous I need to find some $p \in X$ and some open set V of Y with an non-empty intersection with $\Phi(p)$ such that for all $\delta > 0$ there is an $x \in X$ with $|p - x| < \delta$ for which the intersection of $\Phi(x)$ and V is empty.

Proof of upper hemicontinuity

Recall that

$$\Phi(x) = \begin{cases} \{-1\} & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ \{1\} & \text{if } x > 0 \end{cases}$$

To show that Φ is upper hemicontinuous I need to find, for each $p \in X$ and any open superset of $\Phi(p)$ in Y, a $\delta > 0$ such that $|p - x| < \delta$ implies $\Phi(x) \subset V$.

There are three cases: $p < 0$, in which case V is an open set containing -1, or $p = 0$, in which case $V = [-1, 1]$, since that is the only open superset of $[-1, 1]$ in Y, or $p > 0$, in which case V is an open set containing 1.

In the first case $\delta = |p|$ works. $|p - x| < \delta$ implies $x < 0$, which implies $\Phi(x) = \{-1\}$ and therefore $\Phi(x) \subset V$.

In the second case we can choose any positive δ since $V = [-1, 1]$ and hence $\Phi(x) \subset V$ for all x.

In the third case we can again take $\delta = |p|$. The argument is similar to the first case.

Disproof of lower hemicontinuity

Again,

$$\Phi(x) = \begin{cases} \{-1\} & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ \{1\} & \text{if } x > 0 \end{cases}$$

To show that Φ is not lower hemicontinuous I need to find some $p \in X$ and some open set V of Y with an non-empty intersection with $\Phi(p)$ such that for all $\delta > 0$ there is an $x \in X$ with $|p - x| < \delta$ for which the intersection of $\Phi(x)$ and V is empty.

I claim that $p = 0$ and $V = (-1, 1)$ works.

$\Phi(p) = [-1, 1]$ so $\Phi(p) \cap V = (-1, 1)$, which is non-empty.

For any $\delta > 0$ choose $x = \delta/2$.

Then $|p - x| < \delta$ but $\Phi(x) = \{1\}$ has empty intersection with V.

Another example

What about this function?

$$\Phi(x) = \begin{cases} \{-1\} & \text{if } x < 0 \\ \varnothing & \text{if } x = 0 \\ \{1\} & \text{if } x > 0 \end{cases}$$

I claim this function is lower hemicontinuous but not upper hemicontinuous.
To prove that it is lower hemicontinuous we need to show that for every $p \in X$ and every open V in Y which has non-zero intersection with $\Phi(p)$ there is a $\delta > 0$ such that $|p - x| < \delta$ implies that $\Phi(x)$ also has non-empty intersection with V.
Again there are three cases to consider, $p < 0$, $p = 0$, and $p > 0$, and again $\delta = |p|$ works in the first and third cases and any positive δ works in the second case.
To prove that it is not upper hemicontinuous I need to find a p and an open superset V of $\Phi(p)$ in Y such that for every $\delta > 0$ there is some x with $|p - x| < \delta$ such that $\Phi(x)$ is not a subset of V.
In this case $p = 0$ and $V = \varnothing$ works. For any $\delta > 0$ I take $x = \delta/2$.

Hemicontinuity is a local property

If $\Phi\colon X \rightrightarrows Y$ is upper or lower hemicontinuous then the restriction of Φ to any subset S of X is also upper or lower hemicontinuous.

This is obvious from the definitions. Both definitions of hemicontinuity at a point $\mathbf{p} \in X$ require some condition to hold for all $\mathbf{x} \in B_X(\mathbf{p}, \delta)$, either $\Phi(\mathbf{x}) \subset V$ or $\Phi(\mathbf{x}) \cap V \neq \varnothing$, depending on whether it's upper or lower hemicontinuity.

For hemicontinuity of the restriction to S we require this only for all $\mathbf{x} \in B_S(\mathbf{p}, \delta)$, which is a weaker condition, since $S \subset X$ and hence $B_S(\mathbf{p}, \delta) \subset B_X(\mathbf{p}, \delta)$.

Less obviously, if $\mathcal{U}$ is an open cover of X and the restriction of Φ to each $U \in \mathcal{U}$ is hemicontinuous then Φ is hemicontinuous.

Suppose $\mathbf{p} \in X$. Since $\mathcal{U}$ is a cover of X there is some $U \in \mathcal{U}$ such that $\mathbf{p} \in U$. Since it's an open cover this U is open, i.e. there's some $r > 0$ such that $B_X(\mathbf{p}, r) \subset U$.

The restriction of Φ to U is hemicontinuous, so there is a $\delta_U > 0$ such that the appropriate condition holds for all $\mathbf{x} \in B_U(\mathbf{p}, \delta_U)$.

Let $\delta_X = \min\{r, \delta_U\}$. If $\mathbf{x} \in B_X(\mathbf{p}, \delta_X)$ then $\mathbf{x} \in B_X(\mathbf{p}, r)$ so $\mathbf{x} \in U$ and hence $\mathbf{x} \in B_U(\mathbf{p}, \delta_X)$ and therefore $\mathbf{x} \in B_U(\mathbf{p}, \delta_U)$. Our condition therefore holds at $\mathbf{x}$, an arbitrary point in $B_X(\mathbf{p}, \delta_X)$, so Φ is hemicontinuous at $\mathbf{p}$.