MAU34804

Lecture 1

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Recap of closed Leontieff model

Dramatis personae
- a production matrix A, with $a_{ij} \geq 0$ being the amount of good i consumed in the production of good j
- bundles of goods, represented by column vectors with non-negative entries
- sets of prices, also represented by column vectors with non-negative entries

The story so far:
- We considered an equilibrium production problem, with given surpluses.
- We considered a profit maximisation problem, with given prices.

Which industries are profit maximising depends on the prices.

Another problem

Let $\mathbf{q} \gg \mathbf{0}$ be a market basket of goods.
For any set of prices $\mathbf{p} > \mathbf{0}$ we can calculate the total value of the bundle, $\mathbf{p} \cdot \mathbf{q}$.
We can also calculate the fraction of this value represented by the i't good:

$$s_i = \frac{p_i q_i}{\mathbf{p} \cdot \mathbf{q}}.$$

These satisfy the conditions $\sum_{i=1}^{n} s_i = 1$, $s_1 \geq 0, \ldots, s_n \geq 0$.
To produce these goods costs $\mathbf{p} \cdot \mathbf{r}$, where $\mathbf{r} = A\mathbf{q}$.
The fraction of the cost used by making good i is

$$t_i = \frac{q_i \sum_{j=1}^{n} a_{ji} p_j}{\mathbf{p} \cdot \mathbf{r}}.$$

These satisfy the conditions $\sum_{i=1}^{n} t_i = 1$, $t_1 \geq 0, \ldots, t_n \geq 0$.
The t's depend continuously on the p's, and hence on the s's.
We have a continuous function from a simplex to itself. Does it have a fixed point?

Fixed points

Suppose the function on the previous slide has a fixed point, i.e. there is a $\mathbf{p}$ such that $t_i = s_i$ for all i.
Then

$$\frac{p_i q_i}{\mathbf{p}\cdot\mathbf{q}} = s_i = t_i = \frac{q_i \sum_{j=1}^n a_{ji} p_j}{\mathbf{p}\cdot\mathbf{r}}, \qquad \frac{\sum_{j=1}^n a_{ji} p_j}{p_i} = \lambda,$$

where $\lambda = \frac{\mathbf{p}\cdot\mathbf{r}}{\mathbf{p}\cdot\mathbf{q}}$ is independent of i.
$\sum_{j=1}^n a_{ji} p_j = \lambda p_i$ in vector form is just $A^T\mathbf{p} = \lambda\mathbf{p}$, the eigenvector equation.
It didn't really matter which $\mathbf{q}$ we chose, as long as all entries are positive.
Note that if $A^T\mathbf{p} = \lambda\mathbf{p}$ then $\mathbf{p}\cdot\mathbf{r} = \mathbf{p}\cdot A\mathbf{q} = A^T\mathbf{p}\cdot\mathbf{q} = \lambda\mathbf{p}\cdot\mathbf{q}$ so

$$\frac{\mathbf{p}\cdot(\mathbf{q}-\mathbf{r})}{\mathbf{p}\cdot\mathbf{r}} = \frac{1-\lambda}{\lambda}$$

is independent of $\mathbf{q}$, i.e. at prices prices $\mathbf{p}$ the rate of return is the same for all investments.

Topology

Question: Is it true that every continuous function from the simplex $\sum_{i=1}^n s_i = 1$, $s_1 \geq 0 \ldots, s_n \geq 0$ to itself has a fixed point?

The Brouwer fixed point theorem any continuous function from a compact convex subset of Euclidean space to itself has a fixed point. Unfortunately its proof is hard.

We can do the case $n = 2$ without Brouwer. The set is a line segment, so it's enough to prove that any continuous function from a closed interval to itself has a fixed point.

Suppose $f\colon [a, b] \to [a, b]$ is continuous. Let $g(x) = x - f(x)$. Then $g(a) \leq 0$ and $g(b) \geq 0$ so by the intermediate value theorem there is an $x \in [a, b]$ with $g(x) = 0$.

The intermediate value theorem has a constructive proof. We can find the fixed point by the method of bisection.

The Brouwer fixed point theorem has no constructive proof. In higher dimensions there is no algorithm for finding the fixed point.

Brouwer spent most of his career trying, and mostly failing, to convince people not to believe his theorem.

Back to economics

Let's assume the Brouwer fixed point theorem for now. We'll prove it later.

Then for any production matrix $A > O$ there is a price vector $\mathbf{p} > \mathbf{0}$ such that at prices $\mathbf{p}$ production of all goods are equally profitable.

If we strengthen the hypotheses, e.g. to $A \gg O$ then we can show that this vector is unique up to multiplication by a positive scalar.

Taking transposes, there's also a special bundle of goods. I'll leave its economic interpretation as an exercise.

David Wilkins' notes

We'll mostly be following David Wilkins' notes, a copy of which you can find on the module webpage.

Chapter I is a review of multivariable calculus, almost all of which you will have seen. You're responsible for reading that on your own, but I'll highlight the following three theorems:

Theorem 1.2 (Multidimensional Bolzano-Weierstrass Theorem) *Every bounded sequence of points in a Euclidean space has a convergent subsequence.*

Theorem 1.17 (The Multidimensional Extreme Value Theorem) *Let X be a non-empty closed bounded set in $\mathbf{R}^m$, and let $f : X \to \mathbf{R}$ be a continuous real-valued function defined on X. Then there exist points $\mathbf{u}$ and $\mathbf{v}$ of X such that $f(\mathbf{u}) \leq f(\mathbf{x}) \leq f(\mathbf{v})$ for all $x \in X$.*

Theorem 1.21 (The Multidimensional Heine-Borel Theorem) *A subset of n-dimensional Euclidean space $\mathbf{R}^n$ is compact if and only if it is both closed and bounded.*

Functions

Question: What is a function?

According the official Leaving Cert maths curriculum document the concept of a function *involves a set of inputs, a set of possible outputs and a rule that assigns one output to each input.*

This is not how mathematicians use the word function.

Different rules can define the same function, e.g. *add to itself* or *multiply by two*.

Also, there exist functions for which there is no rule defining them.

Question: What is a function, for a mathematician?

A *relation* between two sets X and Y is a subset of the Cartesian product $X \times Y$.

A *function* from X to Y is a relation such that for every $x \in X$ there is a unique $y \in Y$ such that (x, y) satisfies, i.e. is a member of, the relation.

There are plenty of interesting relations which are not functions. For example $x \leq y$ defines a relation between $\mathbf{R}$ and itself.

Functions are important enough that we have a special notation for them. If f is a function then we write $f(x)$ for the unique y mentioned above.

For a mathematician, *functions are graphs*!

Example

Consider a linear function $\sum_{j=0}^{n} c_j s_j$ on the standard n-dimensional simplex

$$\Delta = \left\{ (s_0, \ldots, s_n) \in \mathbf{R}^{n+1} \colon \sum_{j=0}^{n} s_j = 1, s_0 \geq 0, \ldots, s_n \geq 0 \right\}.$$

Really this function *is* the set of pairs $\left((s_0, \ldots, s_n), \sum_{j=0}^{n} c_j s_j \right)$, but usually we don't think of it that way.

For any $(c_0, \ldots, c_n) \in \mathbf{R}^n$ there is a unique maximum value of the corresponding linear function on the simplex Δ. Why?

This follows from the multidimensional extreme value theorem.

So there's a function m from $\mathbf{R}^{n+1}$ to $\mathbf{R}$ such that

$$m((c_0, \ldots, c_n)) = \max_{(s_0, \ldots, s_n) \in \Delta} \sum_{j=0}^{n} c_j s_j.$$

We saw last time that $m((c_0, \ldots, c_n)) = \max\{c_0, \ldots, c_n\}$

Example, continued

With the same set up as on the previous slide we can set up a relation between $\mathbf{R}^{n+1}$ and the simplex Δ as the set of pairs $(((c_0, \ldots, c_n), (s_0, \ldots, s_n))$ for which $(s_0, \ldots, s_n)$ is a maximiser of the function $\sum_{j=0}^n c_j s_j$, i.e. the ones such that

$$\sum_{j=0}^{n} c_j s_j = m((c_0, \ldots, c_n)).$$

A *function* from X to Y is a relation such that for every $x \in X$ there is a unique $y \in Y$ such that (x, y) satisfies, i.e. is a member of, the relation. This relation is not a function!

For each y there is at least one y such that (x, y) satisfies the relation (by the multidimensional extreme value theorem again), but this y isn't unique unless the c's are distinct.

So for talking about maximum problems we need both functions and relations.

Unfortunately people invented a third, entirely unnecessary, notion: correspondences!

Correspondences

For every set there is a power set, the set of its subsets. This is actually one of the axioms of set theory. We denote the power set of X by $\mathcal{P}(X)$.

A *correspondence* from X to Y is a function from X to $\mathcal{P}(Y)$.

For any relation f between sets X and Y we can define a correspondence F from X to Y by saying $F(x) = S$ where S is the set of all $y \in Y$ such that the pair (x, y) satisfies the relation f.

Conversely, for any correspondence F from X to Y we can define a relation f satisfied by those pairs (x, y) such that $y \in F(x)$.

It's easy to see that if we go from a relation to a correspondence and then to a relation in this way we get back the relation we started with, and similarly if go from a correspondence to a relation and then to a correspondence.

You may have heard philosophers say entities must not be multiplied beyond necessity. This is the sort of thing they mean.

Sometimes having two points of view on the same thing is useful, viz functions and graphs. Is that the case for relations and correspondences? Mostly no.

Correspondences are more trouble than they're worth.