MAU34804

Lecture 0

2026-01-19

General module info

MAU 34804 Fixed point theorems and economic equilibria

http://www.maths.tcd.ie/~stalker/34804/

John Stalker (stalker@maths.tcd.ie)

http://www.maths.tcd.ie/~stalker/

Largely follows David Wilkins' syllabus and uses his notes.

No continuous assessment. 100% exam.

A linear production model

The most basic linear production model is the closed Leontieff model.
- Discrete time
- Finitely many goods, say n of them.
- Bundles of goods are represented by column vectors, $\mathbf{q} = (q_1, q_2, \ldots, q_n)$, where q_j is the amount of good j in the bundle.
- Production matrix A, where a_{ij} is the amount of good i consumed in the production of one unit of good j in one unit of time.
- We assume $a_{ij} \geq 0$. We abbreviate this as $A \geq O$. Do not confuse this with positive semidefinite matrices!
- The bundle of goods used in the production of bundle $\mathbf{q}$ is $A\mathbf{q}$, so the *net* production is $\mathbf{q} - A\mathbf{q} = (I - A)\mathbf{q}$.

Question: For a given bundle $\mathbf{s} \geq \mathbf{0}$, can arrange a net production of $\mathbf{s}$ per unit time?
In other words, is there an equilibrium with a net production of $\mathbf{s}$ per unit time?
This happens if and only if there is an $\mathbf{r} \geq \mathbf{0}$ with $\mathbf{r} - A\mathbf{r} = \mathbf{s}$.
The answer will depend on A and $\mathbf{s}$.

Linear Algebra

Question: For a given vector $\mathbf{s} \geq \mathbf{0}$ is there an $\mathbf{r} \geq \mathbf{0}$ with $\mathbf{r} - A\mathbf{r} = \mathbf{s}$?
The answer depends on A and $\mathbf{s}$.
When $n = 1$ the answer is yes if $s_1 = 0$ or $a_{11} < 1$ and no if $s_1 > 0$ and $a_{11} \geq 1$.
Simpler question: Under what conditions on A is it true that for all vectors $\mathbf{s}$ there is a vector $\mathbf{r}$ such that $\mathbf{r} - A\mathbf{r} = \mathbf{s}$?
Answer: If any only if $I - A$ is invertible, i.e. if and only if 1 is not an eigenvalue of A.
What we need though is that for all $\mathbf{s} \geq \mathbf{0}$ there is a $\mathbf{r} \geq \mathbf{0}$.
For this we need $I - A$ to be invertible *and* $(I - A)^{-1} \geq O$.
If all eigenvalues of A have absolute value less than 1 then $\sum_{k=0}^{\infty} A^k$, converges, is the inverse of $I - A$, and is non-negative, so this is a *sufficient* condition. Is it also necessary?
For $n = 1$, yes. What about $n = 2$?

The case $n = 2$

For $n = 2$ the eigenvalues satisfy $\lambda^2 - \operatorname{tr} A + \det A = 0$.
$\operatorname{tr} A = a_{11} + a_{22} \geq 0$. The discriminant, $(\operatorname{tr} A)^2 - 4 \det A$, is

$$(a_{11} + a_{22})^2 - 4(a_{11}a_{22} - a_{12}a_{21}) = (a_{11} - a_{22})^2 + 4a_{12}a_{21} \geq 0$$

so both eigenvalues are real.
Their sum is non-negative so at least one is non-negative and it's at least as large as the other one.
A little algebra shows that

$$(I - A)^{-1} = \frac{1}{(1-\lambda)(1-\mu)} \begin{bmatrix} 1 - a_{22} & a_{12} \\ a_{21} & 1 - a_{11} \end{bmatrix}.$$

If $(I - A)^{-1}$ is non-negative then $1 - \lambda$ and $1 - \mu$ are of the same sign and their sum is non-negative, so both are non-negative, hence positive.
So $1 > \lambda \geq \mu \geq -\lambda > -1$ and all eigenvalues have absolute value less than 1.
In other words, the sufficient condition is also necessary.

Chicken and egg economy

Consider an economy with two goods, chickens and eggs. Producing a chicken requires one egg. Producing an egg requires 1/100 of a chicken. The production matrix is

$$A = \begin{bmatrix} 0 & \frac{1}{100} \\ 1 & 0 \end{bmatrix}.$$

Then

$$(I - A)^{-1} = \begin{bmatrix} \frac{100}{99} & \frac{1}{99} \\ \frac{100}{99} & \frac{100}{99} \end{bmatrix}.$$

To have q_1 chickens and q_2 eggs available for consumption we need to produce $\frac{100q_1+q_2}{99}$ chickens and $\frac{100q_1+100q_2}{99}$ eggs.

If you care, the eigenvalues of A are $\pm - 1/10$.

This is not meant to be a realistic description of the poultry industry.

It is a useful test case for theorems about linear production models. There are published theorems by Nobel prize winners in economics which fail for this example.

Another example

Consider the production matrix

$$A = \begin{bmatrix} \frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & \frac{2}{3} \end{bmatrix}.$$

$$(I - A)^{-1} = \begin{bmatrix} 3 & 3 \\ \frac{9}{2} & \frac{15}{2} \end{bmatrix}.$$

This is a non-negative matrix.
The eigenvalues are $\lambda = \frac{5+\sqrt{29}}{12} = 0.895$ and $\mu = \frac{5-\sqrt{29}}{12} = -0.062$.
This time the maximal eigenvalue is strictly smaller than the other one. This is related to the fact that all entries of A are positive.

Yet another example

Consider the production matrix

$$A = \begin{bmatrix} \frac{1}{5} & \frac{2}{5} \\ \frac{3}{5} & \frac{4}{5} \end{bmatrix}.$$

$$(I - A)^{-1} = \begin{bmatrix} -\frac{5}{2} & -5 \\ -\frac{15}{2} & -10 \end{bmatrix}.$$

Something has clearly gone wrong!

The eigenvalues are $\lambda = \frac{5+\sqrt{29}}{10} = 1.074$ and $\mu = \frac{5-\sqrt{29}}{10} = -0.074$.

$1.074 \geq 1$, which is why things went wrong.

For what it's worth, you don't actually need to find the roots of a polynomial in order to determine whether they're all less than 1 in absolute value. There are various algorithms for doing this, e.g. the Bistritz stability criterion.

Prices

A set of prices determines the value of a bundle of goods. If $\mathbf{p}$ is a vector whose i'th component p_i is the price of the i'th good then the value of the bundle $\mathbf{q}$ is given by the dot product $\mathbf{p} \cdot \mathbf{q}$.

The vector $A^T\mathbf{p}$ gives the production costs of the goods.

The vector $\mathbf{p} - A^T\mathbf{p}$ gives the net profit from producing each good.

Question: Under what conditions on A is it true that for every set of non-negative profits $\mathbf{x} \geq \mathbf{0}$ there is a corresponding set of non-negative prices $\mathbf{p} \geq \mathbf{0}$, i.e. one such that $\mathbf{x} = \mathbf{p} - A^T\mathbf{p}$?

Answer: This is the same linear algebra problem as before, just with A^T in place of A. So it's true if and only if $I - A^T$ is invertible and has a non-negative inverse.

But B is an inverse to $I - A$ if and only if B^T is an inverse to $I - A^T$, and $B^T \geq O$ if and only if $B^T \geq O$, so the conditions are the same as in the previous problem.

In other words, a production matrix allows us to specify arbitrary surplus production targets if and only if it allows us to specify arbitrary unit profits.

Profix maximisation

Suppose we're at an equilibrium with prices $\mathbf{p}$ and you have a budget $b > 0$ to invest. What should you produce?
Making a unit of good i costs y_i and generates a profit of x_i, where $\mathbf{y} = A^T\mathbf{p}$ and $\mathbf{x} = \mathbf{p} - \mathbf{y}$. The return on investment is x_i/y_i.
Let s_i be the fraction of our budget used to produce good i, and let s_0 be the fraction left uninvested. Then we want to maximise the profit

$$b\sum_{i=1}^{n} s_i x_i/y_i$$

subject to the constraints

$$\sum_{i=0}^{n} s_i = 1, \quad s_0 \geq 0, s_1 \geq 0, \cdots, s_n \geq 0.$$

Question: What is the maximum value and where is it attained?

Solution to maximisation problem

The maximum of $\sum_{i=0}^{n} c_i s_i$ subject to the constraints $\sum_{i=0}^{n} s_i = 1, \quad s_0 \geq 0, s_1 \geq 0, \cdots, s_n \geq 0$ is $m = \max_{0 \leq j \leq n} c_j$ and occurs when $s_i = 0$ for any i for which $c_i < m$.

Apply this to

$$c_i = \begin{cases} 0 & \text{if } i = 0 \\ bx_i/y_i & \text{if } i > 0 \end{cases}$$

to get the answer to the profit maximisation problem.

In other words, if some activity has a positive rate of return then spend your full budget doing only that, or only them if there are more than one.

If none have a positive rate of return do nothing, or optionally do any which at least don't have a negative rate of return, if there are any.

Note that the *maximum* depends continuously on the c's, and so on the p's, but *maximiser* doesn't!

It's not even a function, but its graph sort of looks like the graph of a continuous function.