First Order Scalar Equations (Examples)

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1 A linear example

We consider the initial value problem

$$x\frac{\partial u}{\partial x} + (y+1)\frac{\partial u}{\partial y} + u = 0, \quad u(x,0) = f(x).$$

The characteristic equations are

$$\frac{\partial x}{\partial t}(s,t) = x(s,t),$$

$$\frac{\partial y}{\partial t}(s,t) = y(s,t) + 1,$$

$$\frac{\partial u}{\partial t}(s,t) + u(s,t) = 0.$$

The initial conditions are

$$x(s,0) = s, \quad y(s,0) = 0, \quad u(s,0) = f(s).$$

The solution is

$$x = se^t, \quad y = e^t - 1, \quad u = e^{-t}f(s).$$

Eliminating s and t gives

$$u = (y+1)^{-1} f\left(\frac{x}{y+1}\right).$$

2 Another linear example

We consider the initial value problem

$$(1+x^2)\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0, \quad u(x,0) = f(x).$$

The characteristic equations are

$$x' = 1 + x^2, \quad y' = 1, \quad u' = 0.$$

Here the primes denote partial derivatives with respect to t. They are unambiguous because partial derivatives with respect to s don't appear. The initial conditions when $t = 0$ are

$$x = s, \quad y = 0, \quad u = f(s).$$

The solution is

$$x = \frac{s + \tan t}{1 + s \tan t}, \quad y = t, \quad u = f(s).$$

Eliminating s and t gives

$$u = f\left(\frac{x - \tan y}{1 + x \tan y}\right).$$

3 Burgers' Equation

We consider the initial value problem

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0, \quad u(x,0) = f(x).$$

The characteristic equations are

$$t' = 1, \quad x' = u, \quad u' = 0.$$

Derivatives are with respect to τ. We can't use t because that's already taken. Eventually it will turn out that $\tau = t$, but we don't know that initially. The initial conditions at $\tau = 0$ are

$$t = 0, \quad x = s, \quad u = f(s).$$

The solution is

$$t = \tau, \quad x = s + u\tau, \quad u = f(s).$$

Eliminating s and τ gives

$$u = f(x + ut).$$

This equation, by itself, isn't a solution, because the unknown function u appears on both sides. This just gives a relation which u must satisfy.

For a more specific example, consider the initial conditions

$$f(x) = ax^2 + bx + c.$$

The relation

$$u = f(x + ut)$$

then implies

$$u = a(x + ut)^2 + b(x + ut) + c,$$

or

$$at^2u^2 + ((2ax + b)t - 1)u + ax^2 + bx + c = 0.$$

We would like to solve the quadratic and then evaluate at $t = 0$ to identify which root is the correct one, but that's difficult because the leading coefficient vanishes there. So instead we divide by u^2:

$$(ax^2 + bx + c)\left(\frac{1}{u}\right)^2 + ((2ax + b)t - 1)\frac{1}{u} + at^2 = 0.$$

The solution is

$$\frac{1}{u} = \frac{1 - (2ax + b)t + \sqrt{1 - 2(2ax + b)t - (b^2 - 4ac)t^2}}{2(ax^2 + bx + c}.$$

The choice of sign is determined by the fact that

$$\frac{1}{u} = \frac{1}{ax^2 + bx + c}$$

when $t = 0$, or, more simply, by the fact that $1/u \neq 0$. In any case,

$$u = \frac{2(ax^2 + bx + c}{1 - (2ax + b)t + \sqrt{1 - 2(2ax + b)t - (b^2 - 4ac)t^2}}.$$

The solution is of course only valid where

$$1 - 2(2ax + b)t - (b^2 - 4ac)t^2 > 0.$$

That includes all x for $t = 0$, but in general we need

$$2ax + b < \frac{1 - (b^2 - 4ac)t^2}{2t}$$

for $t > 0$ and the reverse inequality for $t < 0$.

4 The Eikonal Equation

We consider the initial value problem

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial 1}\right)^2 - 1 = 0$$

with $u = 0$ on the ellipse

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{4}\right)^2 - 1 = 0.$$

The functions F and H are

$$F(x, y, u, u_x, u_y) = u_x^2 + u_y^2 - 1,$$

$$G(x, y, u, p_x, p_y, p_u) = p_u^2 F\left(x, y, u, -\frac{p_x}{p_u}, -\frac{p_y}{p_u}\right) = p_x^2 + p_y^2 - p_u^2.$$

The characteristic equations are

$$x' = \frac{\partial H}{\partial p_x} = 2p_x, \quad y' = \frac{\partial H}{\partial p_y} = 2p_y, \quad u' = \frac{\partial H}{\partial p_u} = -2p_u,$$

$$p_x' = -\frac{\partial H}{\partial x} = 0, \quad p_y' = -\frac{\partial H}{\partial y} = 0, \quad p_u' = -\frac{\partial H}{\partial u} = 0.$$

We can take the parameterisation of the ellipse to be

$$x = 5\cos s, \quad y = 4\sin s.$$

At those points, $u = 0$, but we also need the derivatives of u, in order to get initial conditions for p_x, p_y and p_u. Differentiating

$$u(5\cos s, 4\sin s) = 0$$

gives

$$-5\sin s\frac{\partial u}{\partial x} + 4\cos s\frac{\partial u}{\partial y} = 0.$$

The solutions to this are all of the form

$$\frac{\partial u}{\partial x} = 4\lambda\cos s\frac{\partial u}{\partial y} = 5\lambda\sin s$$

for some value of λ, which, because

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial 1}\right)^2 - 1 = 0,$$

must satisfy

$$(16\cos^2 s + 25\sin^2 s)\lambda^2 - 1 = 0.$$

So

$$\frac{\partial u}{\partial x} = \pm\frac{4\cos s}{\sqrt{16\cos^2 s + 25\sin^2 s}}, \quad \frac{\partial u}{\partial y} = \pm\frac{5\sin s}{\sqrt{16\cos^2 s + 25\sin^2 s}}.$$

Either choice of sign works, as long as we're consistent. A choice of initial values for p_x, p_y and p_u which satisfies the conditions

$$\frac{\partial u}{\partial x} = -\frac{p_x}{p_u}, \quad \frac{\partial u}{\partial y} = -\frac{p_y}{p_u}$$

is

$$p_x = 4\cos s, \quad p_y = 5\sin s, \quad p_u = \pm\sqrt{16\cos^2 s + 25\sin^2 s}.$$

The solution to this initial value problem for the characteristic equations is

$$x = 5\cos s + 8t\cos s, \quad y = 4\sin s + 10t\sin s,$$

$$u = \mp 2t\sqrt{16\cos^2 s + 25\sin^2 s},$$

$$p_x = 4\cos s, \quad p_y = 5\sin s, \quad p_u = \pm\sqrt{16\cos^2 s + 25\sin^2 s}.$$

From these six equations we need to eliminate the five variables s, t, p_x, p_y and p_u to leave a single equation in the remaining variables x, y and u.

Eliminating p_x, p_y and p_u is easy. We just drop the last three equations, which are the only ones in which they appear. To make the remaining three equations rational we square the last one,

$$u^2 = 64t^2 \cos^2 s + 100t^2 \sin^2 s,$$

and then make the rationalising substitution

$$\cos s = \frac{1 - v^2}{1 + v^2}, \quad \sin s = \frac{2v}{1 + v^2}.$$

The first two equations become

$$x = (5 + 8t)\frac{1 - v^2}{1 + v^2}, \quad y = (4 + 10t)\frac{2t}{1 + v^2},$$

or, multiplying by $1 + v^2$ and rearranging terms,

$$8(1 - v^2)t - x(1 + v^2) + 5(1 - v^2) = 0, \quad 20vt - y(1 + v^2) + 8v = 0.$$

We can write this in matrix form as

$$\begin{pmatrix} 8(1 - v^2) & -x(1 + v^2) + 5(1 - v^2) \\ 20v & -y(1 + v^2) + 8v \end{pmatrix} \begin{pmatrix} t \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

Since the vector

$$\begin{pmatrix} t \\ 1 \end{pmatrix}$$

the matrix

$$\begin{pmatrix} 8(1 - v^2) & -x(1 + v^2) + 5(1 - v^2) \\ 20v & -y(1 + v^2) + 8v \end{pmatrix}$$

must be singular, so its determinant is zero:

$$4(2yv^4 + (5x + 9)v^3 + (5x - 9)v - 2y) = 0.$$

We can get a second equation for v by solving either

$$8(1 - v^2)t - x(1 + v^2) + 5(1 - v^2) = 0$$

or

$$20vt - y(1 + v^2) + 8v = 0$$

for t and substituting into the equation

$$u^2 = 64t^2 \cos^2 s + 100t^2 \sin^2 s,$$

along with the rationalising substitution given earlier. If we use

$$8(1 - v^2)t - x(1 + v^2) + 5(1 - v^2) = 0$$

then, after clearing denominators, we get the equation

$$\begin{aligned}(4x^2 - 4u^2 + 40x + 100)v^8 + (15x^2 + 170x + 225)v^6 \\ +(42x^2 + 8u^2 - 650)v^4 + (15x^2 - 170x + 225)v^2 \\ +4x^2 - 4u^2 - 40x + 100 = 0.\end{aligned}$$

We now have two polynomial equations for v, the one above of order 8 and the one found earlier of order 4.

$$2yv^4 + (5x + 9)v^3 + (5x - 9)v - 2y = 0.$$

For both of these to hold we need their resultant to vanish. The resultant is $(240x)^4$ times the polynomial

$$c_8(x, y)u^8 + c_6(x, y)u^6 + c_4(x, y)u^4 + c_2(x, y)u^2 + c_0(x, y),$$

where

$$\begin{aligned}c_8(x, y) &= 81,\\ c_6(x, y) &= 126x^2 - 612y^2 - 6642,\\ c_4(x, y) &= -239x^4 + 1286x^2y^2 + 1606y^4 - 17874x^2 + 31158y^2 + 200961,\\ c_2(x, y) &= -224x^6 - 1986x^4y^2 - 3462x^2y^4 - 1700y^6 - 20576x^4\\ &\quad - 19516x^2y^2 - 32150y^4 + 498528x^2 - 446850y^2 - 2656800,\\ c_0(x, y) &= (x^4 + 2x^2y^2 + y^4 - 18x^2 + 18y^2 + 81)(16x^2 + 25y^2 - 400)^2\end{aligned}$$

The solution we're looking for clearly doesn't satisfy $240x = 0$, so it must be a root of the polynomial equation

$$c_8(x, y)u^8 + c_6(x, y)u^6 + c_4(x, y)u^4 + c_2(x, y)u^2 + c_0(x, y) = 0.$$

This polynomial is irreducible, so no factorisation is possible. The fact that $c_0(x, y)$ is divisible exactly twice by

$$16x^2 + 25y^2 - 400 = 400\left[\left(\frac{x}{5}\right)^2 + \left(\frac{y}{4}\right)^2 - 1\right]$$

reflects the fact that two of the eight solutions of the polynomial equation are zero on the ellipse

$$\left(\frac{x}{5}\right)^2 + \left(\frac{y}{4}\right)^2 - 1 = 0.$$

These correspond to the two choices of sign in the equations for $\partial u/\partial x$ and $\partial u/\partial y$. Those are the solutions we're looking for. The other six wouldn't satisfy the initial conditions. There isn't really a simpler way to describe the solutions we're looking for. It's possible to use the quartic formula, considering the even polynomial of degree 8 in u as a quartic in u^2, but the result is unenlightening.