Lecture notes
on
Distributions

Hasse Carlsson

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Preface

Two important methods in analysis is differentiation and Fourier transformation. Unfortunally not all functions are differentiable or has a Fourier transform. The theory of distribution tries to remedy this by imbedding classical functions in a larger class of objects, the so called distributions (or general functions). The basic idea is not to think of functions as pointwise defined but rather as a "mean value". A locally integrable function f is identified with the map

$$\varphi \mapsto \int f\varphi,$$

where φ belongs to a space of "nice" test functions, for instance C_0^∞.

As an extension of this we let a distribution be a linear functional on the space of test functions. When extending operations such as differentiation and Fourier transformation, we do this by transfering the operations to the test functions, where they are well defined.

Let us for instance see how to define the derivative of a locally integrable function f on $\mathbb{R}$. If f is continuously differentiable, an integration by parts implies that

$$\int f\varphi = -\int f\varphi'.$$

Now we use this formula to *define* the differential of f, when f is not classically differentiable. f' is the map

$$\varphi \mapsto -\int f\varphi'.$$

In these lectures we will study how differential calculus and Fourier analysis can be extended to distributions and study some applications mainly in the theory of partial differential equations.

The presentation is rather short and for a deeper study I recommend the following books:

Laurent Schwartz. *Théorie des Distributions I, II.* Hermann, Paris, 1950–51.

Lars Hörmander. *The Analysis of Linear Partial Differential Operators I*, 2nd ed. Springer, Berlin, 1990.

Contents

Chapter 1

A primer on C_0^∞-functions

When we shall extend differential calculus to distributions, it is suitable to use infintely differentiable functions with compact support as test functions. In this chapter we will show that there is "a lot of" C_0^∞-functions.

Notation

Let Ω be a domain in $\mathbb{R}^n$. $C^k(\Omega)$ denotes the k times comtinuously differentiable functions on Ω. (k may be $+\infty$.) $C_0^k(\Omega)$ are those functions in $C^k(\Omega)$ with compact support. We denote points in $\mathbb{R}^n$ with $x = (x_1, \dots, x_n)$ and $dx = dx_1 \dots dx_n$ denotes the Lebesgue measure. For a vector $\alpha = (\alpha_1, \dots \alpha_n) \in \mathbb{N}^n$ we let

$$|\alpha| = \alpha_1 + \ldots + \alpha_n, \quad \alpha! = \alpha_1! \ldots \alpha_n!, \quad x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$$

and

$$\partial^\alpha f = \frac{\partial^\alpha f}{\partial x^\alpha} = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}} f.$$

Example 1.1. With these notations the Taylorpolynomial of f of degree N can be written as

$$\sum_{|\alpha| \le N} \frac{\partial^\alpha f(a)}{\alpha!} x^\alpha.$$

□

As described in the preface, to a function $f \in L^1_{\text{loc}}$, we will associate the map Λ_f, given by

$$\varphi \mapsto \int_{\mathbb{R}^n} f \varphi \, dx, \quad \varphi \in C_0^\infty.$$

Problem. Does the map Λ_f determine f?

More precisely, if $f, g \in L^1_{\text{loc}}$ and

$$\int_{\mathbb{R}^n} f\varphi\, dx = \int_{\mathbb{R}^n} g\varphi\, dx, \quad \varphi \in C_0^\infty,$$

does this imply that $f = g$ a.e.? □

To be able to solve this problem we need to construct functions $\varphi \in C_0^\infty$. We start with

Example 1.2. There are functions $f \in C^\infty(\mathbb{R})$ with $f(x) = 0$ when $x \leq 0$ and $f(x) > 0$ when $x > 0$.

Remark 1.3. There is no such real analytic function. □

Proof. Such a function must satisfy $f^{(n)}(0) = 0$ for all n. Thus $f(x) = 0(x^n), x \to 0$, for all n. Guided by this, we put

$$f(x) = \begin{cases} e^{-1/x}, & x > 0 \\ 0, & x \leq 0\,. \end{cases}$$

We have to prove that $f \in C^\infty$. By induction we have

$$f^{(n)}(x) = \begin{cases} P_n\left(\frac{1}{x}\right) e^{-\frac{1}{x}}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

for some polynomials P_n. This is clear when $x \neq 0$. But at the origin we have if $h > 0$,

$$\frac{f^{(n)}(h) - f^{(n)}(0)}{h} = \frac{1}{h} P_n\left(\frac{1}{h}\right) e^{-\frac{1}{h}} \to 0, \quad h \to 0.$$

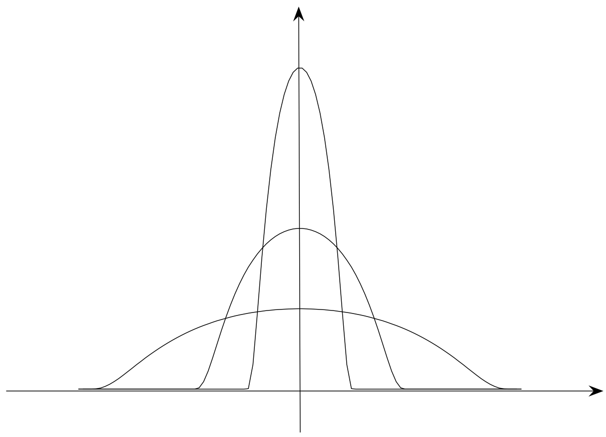
□

Example 1.4. There are non-trivial functions in $C_0^\infty(\mathbb{R}^n)$.

Proof. Let f be the function in Example 2 and put $\varphi(x) = f(1 - |x|^2)$. □

Approximate identities

Pick a function $\varphi \in C_0^\infty(\mathbb{R}^n)$ with $\int \varphi = 1$ and $\varphi \geq 0$. For $\delta > 0$ we let $\varphi_\delta(x) = \delta^{-n}\varphi(x/\delta)$. Then $\varphi_\delta \in C_0^\infty(\mathbb{R}^n)$ and $\int \varphi_\delta = 1$. $\{\varphi_\delta; \delta > 0\}$ is called an approximate identity.



"$\varphi_\delta \to \delta$"

Regularization by convolution

The convolution of two functions f and φ is defined by

$$f * \varphi(x) = \int_{\mathbb{R}^n} f(x-y)\varphi(y)dy.$$

The convolution is defined for instance if $f \in L^1_{\text{loc}}$ and $\varphi \in C_0^\infty$. Then $f * \varphi = \varphi * f, \quad f * \varphi \in C^\infty$ and $\partial^\alpha(f * \varphi) = f * \partial^\alpha \varphi$.

Exercise 1.1. Verify this.

Theorem 1.5.

a) If $f \in C_0$, then $f * \varphi_\delta \to f$, $\delta \to 0$, uniformly.

b) If f is continuous in x, then $f * \varphi_\delta(x) \to f(x)$, $\delta \to 0$.

c) If $f \in L^p$, $1 \leq p < +\infty$, $f * \varphi_\delta \to f$ i L^p (and a.e.).

Remark 1.6. a) implies that $C_0^\infty(\Omega)$ is dense in $C_0(\Omega)$ (in the supremum norm).

Exercise 1.2. Verify this.

Proof.

a) Take R so that supp $\varphi \subset \{x; |x| \le R\}$. We have

$$|f * \varphi_\delta(x) - f(x)| \le \int_{|y|\le\delta R} |f(x-y) - f(x)|\varphi_\delta(y)dy$$
$$\le \text{ uniform continuity } \le \epsilon \int_{\mathbb{R}^n} \varphi_\delta(y)dy = \epsilon, \text{ if } \delta \text{ is small enough.}$$

b) **Exercise 1.3.**

c) Jensen's inequality implies

$$|f * \varphi_\delta(x) - f(x)|^p \le \Big(\int_{\mathbb{R}^n} |f(x-y) - f(x)|\varphi_\delta(y)dy\Big)^p$$
$$\le \int_{\mathbb{R}^n} |f(x-y) - f(x)|^p \varphi_\delta(y)dy = \int_{\mathbb{R}^n} |f(x-\delta t) - f(x)|^p \varphi(t)dt.$$

Using Fubini's theorem and the notation $f^{\delta t}(x) = f(x-\delta t)$, we get

$$\|f * \varphi_\delta - f\|_p^p \le \int_{\mathbb{R}^n} \varphi(t)dt \int_{\mathbb{R}^n} |f(x-\delta t) - f(x)|^p dx$$
$$= \int_{\mathbb{R}^n} \|f^{\delta t} - f\|_p^p \, \varphi(t)dt \to 0,$$

That the limit is zero follows by dominated convergence and that translation is continuous on L^p. This in turn follows since C_0 is dense in L^p, $1 \le p < +\infty$:
If $g \in C_0$, then

$$\|g^\delta - g\|_p^p = \int_K |g(x-\delta) - g(x)|^p dx \to 0, \ \delta \to 0,$$

by dominated convergence. Now approximate $f \in L^p$ with $g \in C_0$, $\|f - g\|_p < \epsilon$. Minkowski's inequality (the triangle inequality) implies

$$\|f^\delta - f\|_p \le \|f^\delta - g^\delta\|_p + \|g^\delta - g\|_p + \|g - f\|_p \le 2\epsilon + \|g^\delta - g\|_p \le 3\epsilon,$$

if δ is small enough. □

Exercise 1.4. a) Let $B_r = \{x; |x| < r\}$. Construct a function $\psi_\delta \in C_0^\infty(\mathbb{R}^n)$ such that $0 \le \psi_\delta \le 1$, $\psi_\delta = 1$ on B_r and supp $\psi_\delta \subset B_{r+\delta}$. How big must $\|\delta^\alpha \psi_\delta\|_\infty$ be?

b) Let $K \subset \Omega$ where K is compact and Ω is open in $\mathbb{R}^n$. Construct $\psi \in C_0^\infty(\Omega)$ with $\psi = 1$ on a neighborhood of K and $0 \le \psi \le 1$. How big must $\|\partial^\alpha \psi\|_\infty$ be?

Now we are able to answer yes to the problem on page 7.

Theorem 1.7. *A locally integrable function that is zero as a distribution is zero a.e.*

Proof. We assume that $\int f\varphi = 0$ for all $\varphi \in C_0^\infty$. According to Theorem 1 a), we have $\int f\Phi = 0$ for all $\Phi \in C_0$, and thus $f = 0$ a.e. (for instance by the Riesz representation theorem.)

Alternatively we can argue as follows: Take $\psi_n \in C_0^\infty$ with $\psi_n(x) = 1$ when $|x| \leq n$. Then $f\psi_n \in L^1$ and

$$f\psi_n * \varphi_\delta(x) = \int_{\mathbb{R}^n} f(y)\psi_n(y)\varphi_\delta(x-y)dy = 0,$$

since $y \mapsto \psi_n(y)\varphi_\delta(x-y)$ is C_0^∞. But $f\psi_n * \varphi_\delta \to f\psi_n$ in L^1 according to Theorem 1 c). Hence $f\psi_n = 0$ a.e., and thus $f = 0$ a.e. □

Chapter 2

Definition of distributions

Definition 2.1. Let Ω be an open domain in $\mathbb{R}^n$. A *distribution* u in Ω is a linear functional on $C_0^\infty(\Omega)$, such that for every compact set $K \subset \Omega$ there are constants C and k such that

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq k} \|\partial^\alpha \varphi\|_\infty, \tag{2.1}$$

for all $\varphi \in C_0^\infty$ with $\operatorname{supp} \varphi \subset K$. $\square$

We denote the distributions on Ω by $\mathscr{D}'(\Omega)$. If the same k can be used for all K, we say that u has order $\leq k$. These distributions are denoted $\mathscr{D}'_k(\Omega)$. The smallest k that can be used is called the order of the distribution. $\mathscr{D}'_F = \cup_k \mathscr{D}'_k$ are the distributions of finite order.

Example 2.2.

(a) A function $f \in L^1_{\text{loc}}$ is a distribution of order 0.

(b) A measure is a distribution of order 0.

(c) $u(\varphi) = \partial^\alpha \varphi(x_0)$ defines a distribution of order $|\alpha|$.

(d) Let x_j be a sequence without limit point in Ω and let

$$u(\varphi) = \sum \partial^{\alpha_j} \varphi(x_j).$$

Then u is a distribution. u has finite order if and only if $\sup |\alpha_j| < \infty$ and then the order is $\sup |\alpha_j|$. $\square$

We will use the notation $\mathscr{D}(\Omega)$ to denote the set $C_0^\infty(\Omega)$, in particular when we consider $\mathscr{D}(\Omega)$ with a topology that corresponds to the the following convergence of test functions.

Definition 2.3. $\varphi_j \to 0$ in $\mathscr{D}(\Omega)$ if, for all j, supp φ_j are contained in a fix compact set and $\|\partial^\alpha \varphi_j\|_\infty \to 0$, $j \to \infty$, for all α. □

Theorem 2.4. *A linear functional u on $\mathscr{D}(\Omega)$ is a distribution if and only if $u(\varphi_j) \to 0$ when $\varphi_j \to 0$ in $\mathscr{D}(\Omega)$.*

Proof. $\Rightarrow$): Trivial.

$\Leftarrow$): Assume that (1) doesn't hold. We have to prove that $u(\varphi_j) \not\to 0$, although $\varphi_j \to 0$ in $\mathscr{D}(\Omega)$. That (1) doesn't hold implies that there is a compact set K and a function $\varphi_j \in \mathscr{D}(\Omega)$, with $\varphi_j \subset K$, $u(\varphi_j) = 1$ and

$$|u(\varphi_j)| > j \sum_{|\alpha| \leq j} \|\partial^\alpha \varphi_j\|_\infty.$$

This implies $\|\partial^\alpha \varphi_j\|_\infty \leq \frac{1}{j}$ if $j \geq |\alpha|$. Thus $\varphi_j \to 0$ in $\mathscr{D}(\Omega)$. □

Theorem 2.5. *A distribution $u \in \mathscr{D}'_k(\Omega)$ can uniquely be extended to a linear functional on $C_0^k(\Omega)$. For every compact set $K \subset \Omega$ there is a constant $C = C_K$ such that*

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq k} \|\partial^\alpha \varphi\|_\infty, \tag{2.2}$$

for all $\varphi \in C_0^k(\Omega)$ with support in K.

Corollary 2.6. *Measures and distributions of order 0 coincides.*

Proof of Theorem 5. Let φ be a fix function in $C_0^k(\Omega)$. Let $\Phi_\delta \in C_0^\infty$ be an approximate identity and put $\varphi_n = \varphi * \Phi_{\frac{1}{n}}$, $n \geq N$. Then all φ_n are supported in a fix compact set K in Ω and if $|\alpha| \leq k$ then

$$\|\partial^\alpha(\varphi - \varphi_n)\|_\infty = \|\partial^\alpha \varphi - (\partial^\alpha \varphi) * \Phi_{\frac{1}{n}}\|_\infty \to 0,\, n \to \infty\,. \tag{2.3}$$

Hence, if u has an extension satisfying (2), then $u(\varphi) = \lim_{n\to\infty} u(\varphi_n)$. This proves the uniqueness of the extension and makes it natural to define

$$u(\varphi) = \lim_{n\to\infty} u(\varphi_n)\,.$$

The limit exists since $u(\varphi_n)$ is a Cauchy sequence:

$$|u(\varphi_n) - u(\varphi_m)| = |u(\varphi_n - \varphi_m)| \leq C \sum_{|\alpha| \leq k} \|\partial^\alpha(\varphi_n - \varphi_m)\| \to 0,$$

as $n, m \to \infty$.

It is easy to see, by taking limits in (1), that u satisfies (2). □

Exercise 2.1. Verify this.

Theorem 2.7. *A positive distribution is a positive measure.*

Definition 2.8. A distribution u is *positive* if $\varphi \geq 0$ implies $u(\varphi) \geq 0$ □

Proof. By Corollary 6 it is enough to show that $u \in \mathscr{D}'_0$.

Assume first that φ is real valued. Let $K \subset\subset \Omega$ and take $\chi \in C_0^\infty(\Omega)$, $0 \leq \chi \leq 1$ with $\chi = 1$ on K. If supp $\varphi \subset K$, then $\chi\|\varphi\|_\infty \pm \varphi \geq 0$. Hence $u(\chi\|\varphi\|_\infty \pm \varphi) \geq 0$, or

$$|u(\varphi)| \leq u(\chi\|\varphi\|_\infty) = u(\chi)\|\varphi\|_\infty.$$

So (1) holds with $k = 0$, $C = u(\chi)$.

If $\varphi = f + ig$ is complex valued,we get

$$|u(\varphi)| \leq |u(f)| + |u(g)| \leq u(\chi)(\|f\|_\infty + \|g\|_\infty) \leq 2u(\chi)\|\varphi\|_\infty.$$

□

Theorem 2.9. *A distribution is determined by its local behavior.*

More precisely: Assume that $\Omega = \cup\Omega_i$ and that $u_i \in \mathscr{D}'(\Omega_i)$. Furthermore we assume that $u_i = u_j$ on $\Omega_i \cap \Omega_j$, i.e. if $\varphi \in C_0^\infty(\Omega_i \cap \Omega_j)$ then $u_i(\varphi) = u_j(\varphi)$. Then there is a unique distribution u on Ω with $u = u_i$ on Ω_i.

To prove this we need a C_0^∞ partition of unity.

Proposition 2.10. *Let $K \subset \cup_1^N \Omega_i$. Then there are $\varphi_i \in C_0^\infty(\Omega_i)$, $0 \leq \varphi_i \leq 1$ and $\Sigma\varphi_i = 1$ on K.*

Proof of Theorem 9. Assume that $u = u_i$ on Ω_i. Let supp $\varphi = K$ and φ_i be a partition av unity as above. By linearity, since $\varphi = \sum_i \varphi\varphi_i$,

$$u(\varphi) = \sum_i u(\varphi\varphi_i) = \sum_i u_i(\varphi\varphi_i) \tag{2.4}$$

This shows the uniqueness.

To prove the existence, we need to show that (4) gives a well defined distribution u. But if $\tilde{\varphi}_k$ is another partition of unity, then $\tilde{\varphi}_k = \sum_i \varphi_i\tilde{\varphi}_k$ on K and thus $\sum_k u_k(\varphi\tilde{\varphi}_k) = \sum_k\sum_i u_k(\varphi\tilde{\varphi}_k\varphi_i) = \sum_i\sum_k u_i(\varphi\tilde{\varphi}_k\varphi_i) = \sum_i u_i(\varphi\varphi_i)$, so (4) defines u uniquely.

It is easy to show that u satisfies (1), and the theorem is proved. □

Exercise 2.2. Do it!

Proof of Proposition 10. We shall show the following
Claim. There are open sets V_i with $\overline{V}_i \subset \Omega_i$ and $K \subset \cup_1^N V_i$.

Assuming this take $\tilde{\varphi}_i \in C_0^\infty(\Omega_i)$, $0 \leq \tilde{\varphi}_i \leq 1$ with $\tilde{\varphi}_i = 1$ on $\overline{V}_i$. Then $\Sigma\tilde{\varphi}_i > 0$ on a neighborhood U of K. Take χ with $\chi = 1$ on K and $\text{supp}\,\chi \subset U$. Put

$$\varphi_i = \chi \frac{\tilde{\varphi}_i}{\Sigma\tilde{\varphi}_i}.$$

It is clear that φ_i satisfy the conditions in the proposition.

To prove the claim, take to $x \in K$ a neighborhood V_x with $x \in V_x \subset \overline{V}_x \subset \Omega_j$ for some j. Then $K \subset \bigcup V_x$. By compactness we get $K \subset \bigcup_1^N V_{x_k}$. Let $V_i = \bigcup_{V_{x_k} \subset \Omega_i} V_{x_k}$. □

The support of a distribution

If $f \in C$ then $\text{supp}\, f = \overline{\{x; f(x) \neq 0\}}$. This implies that $\int f\varphi = 0$ for all $\varphi \in C_0^\infty$ whos support doesn't intersect the support of f.

Definition 2.11. If $u \in \mathscr{D}'(\Omega)$ then $\text{supp}\, u = \{x \in \Omega;$ There is no neighborhood of x with $u = 0$ in this neighborhood.$\}$

Exercise 2.3. Show that supp u is closed.

Theorem 2.12. *If* $\text{supp}\, u \cap \text{supp}\, \varphi = \emptyset$, *then* $u(\varphi) = 0$.

Proof. This follows directly from Theorem 9, since $u = 0$ locally on $\Omega \setminus \text{supp}\, u$. □

An important extension of Theorem 12 is the following theorem and its corollary.

Theorem 2.13. *Assume that* $u \in \mathscr{D}'_k(\Omega)$ *and* $\varphi \in C_0^k(\Omega)$ *with* $\partial^\alpha \varphi(x) = 0$ *if* $|\alpha| \leq k$ *and* $x \in \text{supp}\, u$. *Then* $u(\varphi) = 0$.

Corollary 2.14. *If* $u \in \mathscr{D}'(\Omega)$ *and* $\text{supp}\, u = \{x_0\} \subset \Omega$, *then u is of the form*

$$u(\varphi) = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha \varphi(x_0).$$

Proof of Theorem 13. Let $K = \text{supp}\, u \cap \text{supp}\, \varphi$. If $K = \emptyset$, the result follow from Theorem 5. But K can be non empty. Then, let $K_\epsilon = \{x; d(x, K) < \epsilon\}$

and take $\chi_\epsilon \in C_0^\infty(K_\epsilon)$ with $\chi_\epsilon = 1$ in a neighborhood of K. Then, by Theorem 5,

$$u(\varphi) = u(\chi_\epsilon\varphi + (1-\chi_\epsilon)\varphi) = u(\chi_\epsilon\varphi).$$

If $k=0$ this implies

$$|u(\varphi)| \le C\|\chi_\epsilon\varphi\|_\infty \to 0,\ \epsilon \to 0.$$

If $k>0$ we get

$$|u(\varphi)| \le C\sum_{|\alpha|\le k}\|\partial^\alpha(\chi_\epsilon\varphi)\|_\infty \le C\sum_{|\alpha|+|\beta|\le k}\|\partial^\alpha\chi_\epsilon\partial^\beta\varphi\|_\infty.$$

We can choose χ_ϵ such that $\|\partial^\alpha\chi_\epsilon\|_\infty \le C\epsilon^{-|\alpha|}$. To estimate $\|\partial^\beta\varphi\|_\infty$ we consider the Taylor expansion of φ at a point $x\in K$. Let $y\in K_\epsilon$ and take $x\in K$ with $|x-y|\le\epsilon$. Put

$$g(t) = \partial^\beta\varphi(x+t(y-x)).$$

By the Taylorexpansion of g at $t=0$ of order $k-|\beta|-1$, we get

$$|\partial^\beta\varphi(y)| = |g(1)| = \Big|\sum_{i\le k-|\beta|-1}\frac{g^{(i)}(0)}{i!} + R(y)\Big|.$$

Now $g^{(i)}(0)=0$ and

$$|R(y)| \le C\sup_{0\le s\le 1}|\partial^{k-|\beta|}g(s)| \le C\epsilon^{k-|\beta|}\sum_{|\beta|=k}\|\partial^\beta\varphi\|_{K_\epsilon}.$$

This implies

$$|u(\varphi)| \le C\sum_{|\alpha|+|\beta|\le k}\epsilon^{k-|\alpha|-|\beta|}\sum_{|\beta|=k}\|\partial^\beta\varphi\|_{K_\epsilon} \to 0, \quad \epsilon\to 0.$$

□

Proof of the corollary. u is of finite order k for some k. Fix $\chi\in C_0^\infty(\Omega)$ with $\chi=1$ near x_0 and put

$$\psi(x) = \varphi(x) - \chi(x)\sum_{|\alpha|\le k}(x-x_0)^\alpha\frac{\partial^\alpha\varphi(x_0)}{\alpha!}.$$

Then $\partial^\alpha\psi(x_0)=0$ if $|\alpha|\le k$. By Theorem 13, $u(\psi)=0$ or

$$u(\varphi) = \sum_{|\alpha|\le k}\partial^\alpha\varphi(x_0)u\Big(\frac{(x-x_0)^\alpha}{\alpha!}\chi(x)\Big) = \sum_{|\alpha|\le k}a_\alpha\partial^\alpha\varphi(x_0).$$

□

Exercise 2.4. H 2.2

Exercise 2.5. H 3.1.7.

Exercise 2.6. Show that $u(\varphi) = \sum_1^\infty n^\alpha(\varphi(\frac{1}{n}) - \varphi(-\frac{1}{n}))$ is a distribution of order ≤ 1 if $\alpha < 0$. Also show that supp $u = \{0, \pm 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots\}$, but if K is a closed set with

$$|u(\varphi)| \leq C \sum_{i=0}^{k} \sup_K |\partial^i \varphi|, \quad \varphi \in C_0^\infty(\mathbb{R}) ,$$

then either $\alpha < -1$ or else K contains a neighborhood of the origin. (In particular we can not choose $K =$ supp u.)

Exercise 2.7. Assume that $u \in \mathscr{D}'_k(\mathbb{R})$ and supp $u \subset I$ where I is a compact interval. Show that

$$|u(\varphi)| \leq C \sum_{|\alpha| \leq k} \sup_I |\partial^\alpha \varphi|, \quad \varphi \in C_0^\infty(\mathbb{R}).$$

(Hint. Theorem 13.)

Exercise 2.8. Is there a linear functional u on C_0^∞ that isn't a distribution?

Chapter 3

Operations on distributions

The derivative of distributions

If u is a continuously differentiable function in $\mathbb{R}^n$, an integration by parts gives

$$\int_{\mathbb{R}^n} \partial_k u \cdot \varphi \, dx = -\int_{\mathbb{R}^n} u \cdot \partial_k \varphi \, dx, \quad \varphi \in \mathscr{D},$$

as φ has compact support. This motivates the following definition.

Definition 3.1. If $u \in \mathscr{D}'(\Omega)$, we define $\partial_k u \in \mathscr{D}'(\Omega)$ by

$$\partial_k u(\varphi) = -u(\partial_k \varphi).$$

That $\partial_k u$ defines a distribution follows since

$$|\partial_k u(\varphi)| = |u(\partial_k \varphi)| \leq C \sum_{|\alpha| \leq k} \|\partial^\alpha (\partial_k \varphi)\|_\infty \leq C \sum_{|\alpha| \leq k+1} \|\partial^\alpha \varphi\|_\infty.$$

If $u \in C^1$ the distribution derivative coincides with the classsical derivative.

Example 3.2. Let the Heavisidefunktionen H be defined by

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

Then

$$H'(\varphi) = -H(\varphi') = -\int_0^\infty \varphi'(x) dx = \varphi(0).$$

The Dirac measure at $x_0 \in \mathbb{R}^n$ is given by $\delta_{x_0}(\varphi) = \varphi(x_0)$. So we have showed that $H' = \delta_0$. The derivatives of the Dirac measure are given by $\partial^\alpha \delta_{x_0}(\varphi) = (-1)^{|\alpha|}\delta_{x_0}(\partial^\alpha \varphi) = (-1)^{|\alpha|}\partial^\alpha \varphi(x_0)$. With this notation, by Corollary 2.14, a distribution supported at x_0 can be written as

$$u = \sum_{|\alpha| \le k} C_\alpha \delta_{x_0}^{(\alpha)}.$$

A generalization of Exempel 2 is given by

Proposition 3.3. *Let u be a function in $\Omega \subset \mathbb{R}$, which is continuously differentiable for $x \neq x_0$. Assume that the derivative v is integrable near x_0. Then*

$$u' = v + (u(x_0 + 0) - u(x_0 - 0))\delta_{x_0}.$$

Proof. We start by showing that the limits exist. Let $x_0 < x < y$. Then

$$u(x) = u(y) - \int_x^y v(t)dt.$$

Since v is integrable we obtain as $x \downarrow x_0$

$$u(x_0 + 0) = u(y) - \int_{x_0}^y v(t)dt.$$

By the same argument also $u(x_0 - 0)$ exists. We get

$$\begin{aligned} u'(\varphi) &= -u(\varphi') = -\int_{\mathbb{R}} u\varphi'\,dx = \lim_{\epsilon \to 0} -\int_{|x-x_0|>\epsilon} u(x)\varphi'(x)dx \\ &= \lim_{\epsilon \to 0}\Big\{ -\Big[u(x)\varphi(x)\Big]_{x_0+\epsilon}^{\infty} - \Big[u(x)\varphi(x)\Big]_{-\infty}^{x_0-\epsilon} + \int_{|x-x_0|>\epsilon} v(x)\varphi(x)dx\Big\} \\ &= (u(x_0+0) - u(x_0-0))\varphi(x_0) + \int_{\mathbb{R}} v(x)\varphi(x)dx. \end{aligned}$$

□

Theorem 3.4. *Let u be a distribution on an interval $I \subset \mathbb{R}$. If $u' = 0$, then u is constant.*

Proof. That $u' = 0$ as a distribution means that $u'(\varphi) = 0$ or $u(\varphi') = 0$ for all $\varphi \in \mathscr{D}$. To compute $u(\phi)$, we want to decide if $\phi = \psi'$ for some $\psi \in \mathscr{D}$. This is the case exactly when $\int \phi = 0$ and then $\psi(x) = \int_{-\infty}^x \phi(t)dt$. Thus, if $\int \phi = 0$, then $u(\phi) = 0$. We shall reduce the general case to this special case. Fix $\psi_0 \in \mathscr{D}$ with $\int \psi_0 = 1$. Put $\tilde{\phi} = \phi - \psi_0 \int \phi$. Then $\int \tilde{\phi} = 0$ so $0 = u(\tilde{\phi}) = u(\phi) - u(\psi_0)\int \phi$ or $u(\phi) = u(\psi_0)\int \phi$. Thus u is the constant $u(\psi_0)$. □

Multiplication by functions

$\mathscr{D}(\Omega)$ is a linear space, since we can add distributions and multiply a distribution with a scalar in a natural way. We also want to multiply a distribution with a function f. If u is a locally integrable function, then

$$fu(\varphi) = \int_{\mathbb{R}^n} (fu)\varphi \, dx = \int_{\mathbb{R}^n} u(f\varphi) \, dx = u(f\varphi) \ .$$

To be able to use this to define fu when u is a distribution we need that $f\varphi \in C^\infty$.

Definition 3.5. If $f \in C^\infty$ we define fu by

$$fu(\varphi) = u(f\varphi).$$

Exercise 3.1. Show that $fu \in \mathscr{D}'(\Omega)$.

Remark 3.6. If u is of order k, it is enough to demand that $f \in C^k$. □

Proposition 3.7.
(a) $\partial_j \partial_k u = \partial_k \partial_j u$
(b) $\partial_k (fu) = (\partial_k f)u + f\partial_k u$.

Exercise 3.2. Prove Proposition 7.

Remark 3.8. By (a), the distributional derivatives commutes and we can use the notation $\partial^\alpha u$, $\partial^\alpha u(\varphi) = (-1)^{|\alpha|} u(\partial^\alpha \varphi)$ where α is a multiindex. □

Theorem 3.9. *If $u \in \mathscr{D}'(\Omega)$, $\Omega \subset \mathbb{R}$, and $u' + au = f$ where $f \in C$ and $a \in C^\infty$, Then $u \in C^1$ and the equation is holds classically.*

Proof. Assume first that $a \equiv 0$. Let F be a (classsical) primitive function of f. Then $F \in C^1$ and $(u - F)' = u' - F' = f - f = 0$ as a distribution. Theorem 1 implies that $u = F + C$, and thus $u \in C^1$ and $u' = F' = f$ classically.

If $a \not\equiv 0$, we multiply the equation with its integrating factor. Let A be a primitive function of a. Then A and e^A are C^∞ functions. Furthermore, we have

$$(e^A u)' = e^A u' + e^A a u = e^A (u' + au)$$

in the distributional sense. Therefore, the equation is equivalent to

$$(e^A u)' = e^A f,$$

and we can use the case $a \equiv 0$. □

Exercise 3.3. H 3.1.1

Exercise 3.4. H 3.1.5

Exercise 3.5. H 3.1.14

Exercise 3.6. H 3.1.21

Exercise 3.7. H 3.1.22.

Exercise 3.8. Assume that $u \in \mathscr{D}'(\Omega)$, $\Omega \subset \mathbb{R}$, satisfies $u^{(m)} + a_{m-1}u^{(m-1)} + \ldots + a_0 u = f$, where $f \in C$ and $a_j \in C^\infty$. Show that $u \in C^m$, and that the equation holds classically.

Chapter 4

Finite parts

In this chapter we will extend Proposition 3.3 to the case where the derivative is not locally integrable.

Example 4.1. What is $\frac{d}{dx}\left(\frac{1}{\sqrt{x_+}}\right)$?

We have

$$\begin{aligned}
&\langle (\frac{1}{\sqrt{x_+}})', \varphi\rangle = -\langle \frac{1}{\sqrt{x_+}}, \varphi'\rangle = \\
&= \lim_{\epsilon\to 0} -\int_\epsilon^\infty \frac{1}{\sqrt{x}}\varphi'(x)dx = \lim_{\epsilon\to 0}\Big(-\Big[\frac{1}{\sqrt{x}}\varphi(x)\Big]_\epsilon^\infty - \frac{1}{2}\int_\epsilon^\infty \frac{\varphi(x)}{x^{3/2}}dx\Big) \\
&= -\frac{1}{2}\lim_{\epsilon\to 0}\Big(\int_\epsilon^\infty \frac{1}{x^{3/2}}\varphi(x)dx - \frac{2\varphi(0)}{\sqrt{\epsilon}}\Big)
\end{aligned}$$

□

Definition 4.2.

$$\langle \mathrm{fp}\frac{1}{x_+^{3/2}}, \varphi\rangle = \lim_{\epsilon\to 0}\left\{\int_\epsilon^\infty \frac{\varphi(x)}{x^{3/2}}dx - \frac{2\varphi(0)}{\sqrt{\epsilon}}\right\} .$$

□

Thus we have shown that

$$\frac{d}{dx}(\frac{1}{\sqrt{x_+}}) = -\frac{1}{2}\mathrm{fp}\frac{1}{x_+^{3/2}}.$$

A version of the definition that is easier to remember is

$$\langle \mathrm{fp}\frac{1}{x_+^{3/2}}, \varphi\rangle = \int_0^\infty \frac{\varphi(x)-\varphi(0)}{x^{3/2}}dx.$$

Example 4.3. We define $\text{fp}\dfrac{1}{|x|^{5/2}}$ by

$$\langle \text{fp}\frac{1}{|x|^{5/2}}, \varphi\rangle = \int_{\mathbb{R}} \frac{\varphi(x) - \varphi(0) - x\varphi'(0)}{|x|^{5/2}} dx \ .$$

The order of $\text{fp}\dfrac{1}{|x|^{5/2}}$ is 2. To show this we split the integral into two pieces,

$$\langle \text{fp}\frac{1}{|x|^{5/2}}, \varphi\rangle = \int_{|x|\le 1} + \int_{|x|>1} \frac{\varphi(x) - \varphi(0) - x\varphi'(0)}{|x|^{5/2}} dx = \text{I+II}.$$

To estimate the first integral we use that

$$|\varphi(x) - \varphi(0) - x\varphi'(0)| \le \frac{1}{2} x^2 \|\varphi''\|_\infty \ .$$

This implies

$$|\text{I}| \le \frac{1}{2}\|\varphi''\|_\infty \int_{|x|\le 1} \frac{1}{|x|^{1/2}} dx \le C\|\varphi''\|_\infty \ .$$

For the second integral we have

$$|\text{II}| \le \int_{|x|>1} \frac{2\|\varphi\|_\infty + |x|\|\varphi'\|_\infty}{|x|^{5/2}} dx \le C(\|\varphi\|_\infty + \|\varphi'\|_\infty) \ .$$

Thus the order is at most two.

To show that the order can not be smaller, we let $\varphi \in C_0^\infty$, $0 \le \varphi \le 1$, $\text{supp}\, \varphi \subset (0,3)$ and $\varphi = 1$ on $[1,2]$ and put $\varphi_\epsilon(x) = \varphi(x/\epsilon)$. Then

$$|\langle \text{fp}\frac{1}{|x|^{5/2}}, \varphi_\epsilon\rangle| = \int_{\mathbb{R}} \frac{\varphi_\epsilon(x)}{x^{5/2}} dx \ge \int_\epsilon^{2\epsilon} \frac{1}{x^{5/2}} dx \ge c\frac{1}{\epsilon^{3/2}} \ .$$

Furthermore $\|\varphi_\epsilon\|_\infty + \|\varphi_\epsilon'\|_\infty \le C/\epsilon$. Thus if the order were less than 2, we would have $c/\epsilon^{3/2} \le |\langle \text{fp}\dfrac{1}{|x|^{5/2}}, \varphi\rangle| \le C/\epsilon$, a contradiction.

Since the order of $\text{fp}\dfrac{1}{|x|^{5/2}}$ is 2 and $|x|^{5/2} \in C^2$, $|x|^{5/2}\text{fp}\dfrac{1}{|x|^{5/2}}$ is well defined and

$$\langle |x|^{5/2}\text{fp}\frac{1}{|x|^{5/2}}, \varphi\rangle = \langle \text{fp}\frac{1}{|x|^{5/2}}, |x|^{5/2}\varphi\rangle$$
$$= \int_{\mathbb{R}} \frac{|x|^{5/2}\varphi(x)}{|x|^{5/2}} dx = \int_{\mathbb{R}} \varphi(x) dx = \langle 1, \varphi\rangle.$$

Here we have used that $|x|^{5/2}\varphi(x)$ and its derivative vanishes at $x = 0$.

Thus $\text{fp}\frac{1}{|x|^{5/2}}$ solves the division problem $|x|^{5/2}u = 1$. □

Exercise 4.1. Show that $(\text{fp}\frac{1}{x_+^{3/2}})' = -\frac{3}{2}\text{fp}\frac{1}{x_+^{5/2}}$.

The above examples can be generalized to to define fp x_+^{-a}, fp $|x|^{-a}$ and (for certain a) fp x^{-a} etc. when a *is not* an integer, for instance

$$\langle \text{fp}\frac{1}{|x|^a}, \varphi\rangle = \int_{-\infty}^{\infty} \frac{\varphi(x) - P(x)}{|x|^a} dx,$$

where P is the Taylorpolynomial of φ at the origin of order [a] - 1. Then

$$(\text{fp}\frac{1}{|x|^a})' = -a\,\text{fp}\frac{\text{sgn}\, x}{|x|^{a+1}} = a\left(\text{fp}\frac{1}{x_+^{a+1}} - \text{fp}\frac{1}{x_-^{a+1}}\right)$$

och

$$|x|^a \text{fp}\frac{1}{|x|^a} = 1.$$

Another important property of $\text{fp}\frac{1}{|x|^a}, a \neq -1, -2, \ldots$, is that it is homogeneous of degree $-a$. As we shall see later this fact simplifies the computation of its Fourier transform.

To show that $\text{fp}\frac{1}{|x|^a}$ is homogeneous we first must define what this means. If $u(x)$ is a function on $\mathbb{R}^n$, u is homogeneous of degree α if $u(tx) = t^\alpha u(x)$, $t > 0$. This can be reformulated in a way that is meaningful for distributions. For a function u, we have

$$\langle u(tx), \varphi\rangle = \int_{\mathbb{R}} u(tx)\varphi(x)dx = [y = tx] = \int_{\mathbb{R}} u(y)\frac{1}{t^n}\varphi(\frac{y}{t})dy = \langle u, \varphi_t\rangle.$$

But if u is homogeneous of degree α, we also have

$$\langle u(tx), \varphi\rangle = \int_{\mathbb{R}} u(tx)\varphi(x)dx = \int_{\mathbb{R}} t^\alpha u(x)\varphi(x)dx = t^\alpha \langle u, \varphi\rangle.$$

Therefore we make the following definition.

Definition 4.4. $u \in \mathscr{D}'(\mathbb{R}^n)$ is *homogeneous of degree* α if

$$\langle u, \varphi_t\rangle = t^\alpha \langle u, \varphi\rangle, t > 0.$$

□

Proposition 4.5. $\text{fp}\frac{1}{|x|^a}$ *och* $\text{fp}\frac{1}{x_+^a}$ *are homogeneous of degree* $-a$ *if* $a \neq 1, 2, 3, \ldots$

Proof when $a = \frac{5}{2}$. We have $\varphi_t(0) = \frac{1}{t}\varphi(0)$ and $\varphi_t'(0) = \frac{1}{t^2}\varphi'(0)$. Thus

$$\langle \mathrm{fp}\frac{1}{|x|^{5/2}}, \varphi_t\rangle = \int_{\mathbb{R}} \frac{1}{|x|^{5/2}}\Big(\frac{1}{t}\varphi(\frac{x}{t}) - \frac{1}{t}\varphi(0) - \frac{x}{t^2}\varphi(0)\Big)dx$$
$$= \Big[y = \frac{x}{t}\Big] = \int_{\mathbb{R}} \frac{1}{t^{5/2}|x|^{5/2}}(\varphi(x) - \varphi(0) - x\varphi'(0))dx = \frac{1}{t^{5/2}}\langle \mathrm{fp}\frac{1}{|x|^{5/2}}, \varphi\rangle.$$

□

Example 4.6. Compute $(\log|x|)'$.

We have

$$\langle(\log|x|)', \varphi)\rangle = -\langle \log|x|, \varphi'\rangle = -\int_{\mathbb{R}} \varphi'(x)\log|x|dx$$
$$= -\lim_{\epsilon\to 0}\int_{|x|>\epsilon} \varphi'(x)\log|x|dx = \lim_{\epsilon\to 0} -\Big\{[\varphi(x)\log|x|]_{-\infty}^{-\epsilon}$$
$$+ [\varphi(x)\log|x|]_{\epsilon}^{\infty} - \int_{|x|>\epsilon}\varphi(x)\frac{dx}{x}\Big\}$$
$$= \lim_{\epsilon\to 0}\Big\{\int_{|x|>\epsilon}\varphi(x)\frac{dx}{x} + (\varphi(\epsilon) - \varphi(-\epsilon))\log\epsilon\Big\}$$
$$= \lim_{\epsilon\to 0}\int_{|x|>\epsilon}\varphi(x)\frac{dx}{x} = \mathrm{pv}\int_{\mathbb{R}}\frac{\varphi(x)}{x}dx\ ,$$

where the last equality is a definition. □

Definition 4.7. $\langle \mathrm{pv}\frac{1}{x}, \varphi\rangle = \lim_{\epsilon\to 0}\int_{|x|>\epsilon}\frac{\varphi(x)}{x}dx.$

If we instead differentiate $\log x_+$, we get

$$\langle \log x_+, \varphi\rangle = \lim_{\epsilon\to 0} - \int_{\epsilon}^{\infty}\varphi'(x)\log x dx =$$
$$= \lim_{\epsilon\to 0}\left(-[\varphi(x)\log x]_{\epsilon}^{\infty} + \int_{\epsilon}^{\infty}\frac{\varphi(x)}{x}dx\right) =$$
$$= \lim_{\epsilon\to 0}\left(\int_{\epsilon}^{\infty}\frac{\varphi(x)}{x}dx + \varphi(0)\log\epsilon\right) = \lim_{\epsilon\to 0}\left(\int_{0}^{\infty}\frac{\varphi(x)}{x}dx - \int_{\epsilon}^{1}\frac{\varphi(0)}{x}dx\right) =$$
$$= \int_0^{\infty}\frac{\varphi(x) - \chi(x)\varphi(0)}{x}dx,$$

where $\chi = \chi_{[-1,1]}$, as in the rest of this chapter.

Thus with the following

Definition 4.8. $\langle \mathrm{fp}\frac{1}{x_+}, \varphi \rangle = \int_0^\infty \frac{\varphi(x) - \chi(x)\varphi(0)}{x} dx$ □

we have proved that $(\log x_+)' = \mathrm{fp}\frac{1}{x_+}$.

Exercise 4.2. Show that $\mathrm{fp}\frac{1}{x_+}$ solves the division problem $xu = H$.

The above examples can be generalized to the following

Definition 4.9.

$$\langle \mathrm{fp}\frac{1}{|x|^n}, \varphi \rangle = \int_{-\infty}^\infty \frac{\varphi(x) - P(x) - \frac{x^{n-1}}{(n-1)!}\varphi^{(n-1)}(0)\chi(x)}{|x|^n} dx$$

if $n = 1, 2, 3, \ldots$ and P is the Taylorpolynomial of φ of degree $n-2$. □

Example 4.10. $(\mathrm{fp}\frac{1}{x_+^2})' = -2\mathrm{fp}\frac{1}{x_+^3} + \frac{1}{2}\delta^{(2)}$. □

Proof.

$$\langle (\mathrm{fp}\frac{1}{x_+^2})', \varphi \rangle = -\langle \mathrm{fp}\frac{1}{x_+^2}, \varphi' \rangle = -\int_0^\infty \frac{\varphi'(x) - \varphi'(0) - x\varphi''(0)\chi(x)}{x^2} dx =$$
$$-\lim_{\epsilon \to 0} \left\{ \int_\epsilon^\infty \frac{\varphi'(x) - \varphi'(0)}{x^2} dx - \int_\epsilon^1 \frac{x\varphi''(0)}{x^2} dx \right\} .$$

As $\varphi(x) - \varphi(0) - x\varphi'(0)$ is a primitive function of $\varphi'(x) - \varphi'(0)$, an integration by parts in the first integral implies that

$$\langle (\mathrm{fp}\frac{1}{x_+^2})', \varphi \rangle = -\lim_{\epsilon \to 0} \left\{ \left[\frac{\varphi(x) - \varphi(0) - x\varphi'(0)}{x^2} \right]_\epsilon^\infty + \right.$$
$$\left. 2\int_\epsilon^\infty \frac{\varphi(x) - \varphi(0) - x\varphi'(0)}{x^3} dx - \int_\epsilon^1 \frac{x\varphi''(0)}{x^2} dx \right\} .$$

Now

$$\lim_{\epsilon \to 0} \left[\frac{\varphi(x) - \varphi(0) - x\varphi'(0)}{x^2} \right]_\epsilon^\infty = -\frac{1}{2}\varphi''(0)$$

and

$$\lim_{\epsilon \to 0} \left\{ 2\int_\epsilon^\infty \frac{\varphi(x) - \varphi(0) - x\varphi'(0)}{x^3} dx - \int_\epsilon^1 \frac{x\varphi''(0)}{x^2} dx \right\} =$$
$$2\int_0^\infty \frac{\varphi(x) - \varphi(0) - x\varphi'(0) - \frac{1}{2}x^2\varphi''(0)\chi(x)}{x^3} dx .$$

Hence

$$\langle (\mathrm{fp}\frac{1}{x_+^2})', \varphi\rangle = \frac{1}{2}\varphi''(0) - 2\int_0^\infty \frac{\varphi(x) - \varphi(0) - x\varphi'(0) - \frac{1}{2}x^2\varphi''(0)\chi(x)}{x^3}\, dx$$
$$= \langle -2\mathrm{fp}\frac{1}{x_+^3} + \frac{1}{2}\delta^{(2)}, \varphi\rangle.$$

□

Example 4.11. $\mathrm{fp}\frac{1}{|x|^3}$ is *not* homogeneous of degree -3 since

$$\langle \mathrm{fp}\frac{1}{|x|^3}, \varphi_t\rangle = \int_{\mathbb{R}} \frac{\frac{1}{t}\varphi(\frac{x}{t}) - \frac{1}{t}\varphi(0) - \frac{x}{t^2}\varphi'(0) - \frac{1}{2}\frac{x^2}{t^3}\varphi''(0)\chi(x)}{|x|^3} dx$$
$$= \left[y = \frac{x}{t}\right] = \frac{1}{t^3}\int_{\mathbb{R}} \frac{\varphi(x) - \varphi(0) - x\varphi'(0) - \frac{1}{2}x^2\varphi''(0)\chi(xt)}{|x|^3} dx$$
$$= (\text{If we assume that } t > 1) = \frac{1}{t^3}\langle \mathrm{fp}\frac{1}{|x|^3}, \varphi\rangle + \frac{1}{2t^3}\varphi''(0)\int_{\frac{1}{t}<|x|<1} \frac{dx}{|x|}$$
$$= \frac{1}{t^3}\langle \mathrm{fp}\frac{1}{|x|^3}, \varphi\rangle + \varphi''(0)\frac{\log t}{t^3}.$$

□

Exercise 4.3. What happens if $t < 1$?

Exercise 4.4. Is $\mathrm{fp}\frac{1}{x^3}$ homogeneous of degree -3?

Exercise 4.5. Show that the equation $x^N u = 0$ has the solution $u = \sum_{n=0}^{N-1} c_n\delta^{(n)}$.

Since $x^N\mathrm{fp}\frac{1}{x^N} = 1$, Exercise 4.5 implies that the equation

$$x^N u = 1 \text{ has the general solution } u = \mathrm{fp}\frac{1}{x^N} + \sum_{n=0}^{N-1} c_n\delta^{(n)} .$$

In the same way the equation

$$(x-a)^N u = 1 \text{ has the solution } u = \mathrm{fp}\frac{1}{(x-a)^N} + \sum_{n=0}^{N-1} c_n\delta_a^{(n)} .$$

where $\mathrm{fp}\frac{1}{(x-a)^N}$ is defined in the same way as $\mathrm{fp}\frac{1}{x^N}$.

Now we can solve the division problem $Pu = 1$, where P is a polynomial of one variable. In a neighborhood where $P \neq 0$, $u = 1/P$ is a nice function. So the the only problem is to understand $1/P$ near a real zero a of P. But there we have $P(x) = (x-a)^n Q(x)$ where $Q(a) \neq 0$. Hence, near $x = a$, we have $(x-a)^n Q(x) u = 1$. This is satisfied if $Qu = \operatorname{fp} \frac{1}{(x-a)^n}$. Hence $u = \frac{1}{Q(x)} \operatorname{fp} \frac{1}{(x-a)^n}$ solves $Pu = 1$ near $x = a$. By Theorem 2.9, u is a well defined distribution on $\mathbb{R}$ that solves $Pu = 1$.

Exercise 4.6. H 3.1.14

Exercise 4.7. H 3.1.20

Exercise 4.8. Let u be a continuous function on $\mathbb{R}^n \setminus \{0\}$ that is homogeneous of degree $-n$. Show that we can define a distribution $\operatorname{pv} u$ by

$$\langle \operatorname{pv} u, \varphi \rangle = \lim_{\epsilon \to 0} \int_{|x| > \epsilon} u(x)\varphi(x)dx,$$

if and only if $\int_{|x|=1} u(x) d\sigma(x) = 0$.

Exercise 4.9. An alternative method to define $\operatorname{fp} x_+^\alpha$ is by analytic continuation. If $\varphi \in \mathscr{D}$ and $\operatorname{Re} \alpha > -1$ the map

$$F_\varphi(\alpha) = \int_0^\infty x^\alpha \varphi(x) dx$$

is analytic. Show that this map can be continued to a meromophic function in $\mathbb{C}$, whose only singularities are simple poles in $-1, -2, -3, \ldots$. Compute the residues R_{-k} of F_φ and show that if we for $k = 1, 2, 3, \ldots$ extend the definition of F_φ by

$$F_\varphi(-k) = \lim_{\alpha \to -k} \left(F_\varphi(\alpha) - \frac{R_{-k}}{\alpha + k} \right),$$

we have

$$F_\varphi(\alpha) = \langle \operatorname{fp} x_+^\alpha, \varphi \rangle.$$

This approach gives an alternative proof that

$$x_+^\alpha \operatorname{fp} x_+^{-\alpha} = H, \quad \alpha \in \mathbb{C}$$

and

$$(\operatorname{fp} x_+^\alpha)' = \alpha \operatorname{fp} x_+^{\alpha - 1}, \quad \alpha \neq 0, -1, -2, -3, \ldots$$

Chapter 5

Fundamental solutions of the Laplace and heat equations

Definition 5.1. Let $P(D)$ be a differential operator. A distribution E with $P(D)E = \delta$, is called a *fundamental solution* of P.

$\square$

In Example 3.2, we saw that the Heaviside function H is a fundamental solution of d/dx. A little more general, $H(x_1)\dots H(x_n)$ is a fundamental solution of $\partial_1 \dots \partial_n$.

In this chapter we will treat the Laplace operator

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2},$$

and the heat operator

$$\left(\frac{\partial}{\partial t} - \Delta_x\right) u = \frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2}.$$

To accomplish this we need to be able to integrate by parts in $\mathbb{R}^n$, and we remind the reader about

Green's identity

$$\int_\Omega (u\Delta v - v\Delta u)\, dx = \int_{\partial\Omega} \left(u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}\right) d\sigma$$

where $\partial/\partial n$ is the exterior normal derivative.

Theorem 5.2.

$$E(x) = \begin{cases} \frac{1}{2\pi} \log|x|, & n = 2, \\ -\frac{1}{\omega_n (n-2)|x|^{n-2}}, & n \geq 3, \end{cases}$$

is a fundamental solution of the Laplace operator in $\mathbb{R}^n$.
(ω_n is the surface measure of the unit sphere in $\mathbb{R}^n$.)

Exercise 5.1. Compute ω_n in terms of the Γ function,

$$\Gamma(s) = \int_0^\infty t^{s-1} e^{-t} dt, \ \mathrm{Re} s > -1.$$

Exercise 5.2. Visa att $\Delta E(x) = 0$ om $x \neq 0$.

Proof.

$$\langle \Delta E, \varphi \rangle = \langle E, \Delta \varphi \rangle = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} E \Delta \varphi dx$$

$$= \text{Exercise 2} = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} (E \Delta \varphi - \varphi \Delta E) dx = \text{Green's identity} =$$

$$= \lim_{\epsilon \to 0} \int_{|x|=\epsilon} \left(E \frac{\partial \varphi}{\partial n} - \varphi \frac{\partial E}{\partial n} \right) d\sigma = \lim_{\epsilon \to 0} (I_\epsilon + II_\epsilon).$$

We only consider the case $n \geq 3$, and leave the case $n = 2$ as
Exercise 5.3.

We have

$$|I_\epsilon| \leq C \left\| \frac{\partial \varphi}{\partial n} \right\|_\infty \frac{1}{\epsilon^{n-2}} \omega_n \epsilon^{n-1} \longrightarrow 0, \epsilon \to 0 \ ,$$

and as $\partial / \partial n = - \partial / \partial r$,

$$II_\epsilon = \int_{|x|=\epsilon} \varphi \frac{\partial E}{\partial r} d\sigma = - \frac{1}{(n-2)\omega_n} \int_{|x|=\epsilon} \varphi(x) \frac{-(n-2)}{|x|^{n-1}} d\sigma(x)$$

$$= \frac{\varphi(0)}{\omega_n} \int_{|x|=\epsilon} \frac{d\sigma(x)}{\epsilon^{n-1}} + \frac{1}{\omega_n} \int_{|x|=\epsilon} (\varphi(x) - \varphi(0)) \frac{d\sigma(x)}{\epsilon^{n-1}}$$

$$\longrightarrow \varphi(0), \epsilon \to 0.$$

□

Theorem 5.3.

$$E(x,t) = \begin{cases} \frac{1}{(4\pi t)^{n/2}} \exp(-\frac{|x|^2}{4t}), & t > 0, \\ 0, & t < 0, \end{cases}$$

is a fundamental solution of the heat equation in $\mathbb{R}^{n+1}$.

Exercise 5.4. Show that $(\frac{\partial}{\partial t} - \Delta_x)E(t,x) = 0$ if $t \neq 0$.

Proof. Let $\phi(x) = E(x, \frac{1}{2})$. When $n = 1$, this is the density of a $N(0,1)$ distributed stochastic variable and, when $n > 1$ the product of n such densities. Furthermore, $E(x,t) = \phi_{\sqrt{2t}}(x)$ and thus $\int_{\mathbb{R}^n} E(x,t)dx = 1$ for all $t > 0$ and $E \in L^1_{loc}(\mathbb{R}^{n+1})$. Now

$$\left\langle \frac{\partial E}{\partial t}, \varphi \right\rangle = -\left\langle E, \frac{\partial \varphi}{\partial t} \right\rangle = \lim_{\epsilon \to 0} - \int_{\mathbb{R}^n} dx \int_{t > \epsilon} E \frac{\partial \varphi}{\partial t} dt$$
$$= \lim_{\epsilon \to 0} \left\{ \int_{\mathbb{R}^n} E(x,\epsilon)\varphi(x,\epsilon)dx + \int_{\mathbb{R}^n} \int_{t>\epsilon} \varphi \frac{\partial E}{\partial t} dxdt \right\},$$

and

$$\langle \Delta_x E, \varphi \rangle = \langle E, \Delta_x \varphi \rangle = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \int_{t>\epsilon} E \Delta_x \varphi dxdt = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} \int_{t>\epsilon} \Delta_x E \varphi dxdt$$

Thus

$$\left\langle \left(\frac{\partial}{\partial t} - \Delta_x \right) E, \varphi \right\rangle =$$
$$\lim_{\epsilon \to 0} \left\{ \int_{\mathbb{R}^n} E(x,\epsilon)\varphi(x,\epsilon)dx + \int_{\mathbb{R}^n} \int_{t>\epsilon} \varphi \left(\frac{\partial E}{\partial t} - \Delta_x E \right) dxdt \right\} =$$
$$= \text{Exercise 4} = \lim_{\epsilon \to 0} \int_{\mathbb{R}^n} E(x,\epsilon)\varphi(x,\epsilon)dx.$$

Since $E(x,t) = \phi_{\sqrt{2t}}(x)$ is an approximate identity, we ought to have

$$I_\epsilon = \int_{\mathbb{R}^n} E(x,\epsilon)\varphi(x,\epsilon)dx \to \varphi(0),\ \epsilon \to 0.$$

This does not follow directly from Theorem 1.4 since the support of $\varphi(x,\epsilon)$ is not compact and depend on ϵ. But the change of variables $x = \sqrt{2\epsilon}\, y$ gives

$$I_\epsilon = \int_{\mathbb{R}^n} \phi(x)\varphi(\sqrt{2\epsilon}\, x, \epsilon)dx\ .$$

Since $\phi \in L^1$ and $|\varphi(\sqrt{2\epsilon}\, x, \epsilon)| \leq \|\varphi\|_\infty$, we get by dominated convergence

$$\lim_{\epsilon \to 0} I_\epsilon = \int_{\mathbb{R}^n} \phi(x) \lim_{\epsilon \to 0} \varphi(\sqrt{2\epsilon}\, x, \epsilon)dx = \int_{\mathbb{R}^n} \phi(x)\varphi(0,0)dx = \varphi(0)\ .$$

□

Exercise 5.5. Show that $\dfrac{1}{\pi z}$ is a fundamental solution to $\dfrac{\partial}{\partial \bar{z}}$ in $\mathbb{C}$.

Exercise 5.6. Compute $\dfrac{\partial}{\partial z} \log|z|$ and $\Delta \log|z|$ in $\mathbb{C}$.

Exercise 5.7. H 3.3.9

Exercise 5.8. H 3.3.11

Exercise 5.9. H 3.3.12

Chapter 6

Distributions with compact support

Theorem 6.1. *Assume that $u \in \mathscr{D}'(\Omega)$ has compact support Then u has a unique extension to $C^\infty(\Omega)$ that satisfies $u(\varphi) = 0$ if $\operatorname{supp} u$ and $\operatorname{supp}\varphi$ are disjoint.*

If K is a compact set that contains a neighborhood of $\operatorname{supp} u$, then

$$|u(\varphi)| \leq C \sum_{|\alpha|\leq k} \|\partial^\alpha \varphi\|_K, \quad \varphi \in C^\infty(\Omega). \tag{6.1}$$

Proof. Take $\chi \in C_0^\infty(K)$ with $\chi = 1$ in a neighborhood of $\operatorname{supp} u$. If $\varphi \in C_0^\infty$, then according to Theorem 2.12

$$u(\varphi) = u(\chi\varphi + (1-\chi)\varphi) = u(\chi\varphi) + u((1-\chi)\varphi) = u(\chi\varphi).$$

Thus

$$u(\varphi) = u(\chi\varphi),\ \varphi \in C^\infty$$

defines an extension of u. (1) follows Leibnitz' rule.

Assume on the other hand that u_1 is an extension to C^∞. The condition on the support implies that $u_1((1-\chi)\varphi) = 0$, and consequently $u_1(\varphi) = u_1(\chi\varphi) = u(\chi\varphi)$ and thus the extension is unique.

□

Remark 6.2. Exercise 2.6 shows that it is not always possible to take $K = \operatorname{supp} u$ in (1). □

Exercise 6.10. State and prove a converse of Theorem 1.

Thus we can identify distributions with compact support with the linear functionals on $C^\infty(\Omega)$ that satisfies (1). These distributions are denoted $\mathscr{E}'(\Omega)$.

Chapter 7

Convergence of distributions

Definition 7.1. A sequence $u_j \in \mathscr{D}'(\Omega)$ converges to $u \in \mathscr{D}'(\Omega)$ if

$$u_j(\varphi) \to u(\varphi), j \to \infty,$$

for every test function $\varphi \in \mathscr{D}(\Omega)$. We denote this by $u_j \to u$ in $\mathscr{D}'$. □

If $u_j \to u$ in $\mathscr{D}'$, we also have $\partial^\alpha u_j \to \partial^\alpha u$ in $\mathscr{D}'$ for every multiindex α. We write $u = \sum u_j$ in $\mathscr{D}'$ if the partial sums converges in $\mathscr{D}'$. If the series converges, it is differentiable and we have $\partial^\alpha(\sum u_j) = \sum \partial^\alpha u_j$.

Remark 7.2. Convergence in $\mathscr{D}'$ is a "weak" condition, if for instance $f_j \to f$ in L^p then $f_j \to f$ i $\mathscr{D}'$. □

Exercise 7.1. Prove that.

Definition 7.3. $u_j \in \mathscr{D}'(\Omega)$ is a Cauchy sequence in $\mathscr{D}'(\Omega)$ if $u_j(\varphi)$ is a Cauchy sequence in $\mathbb{C}$ for every $\varphi \in \mathscr{D}(\Omega)$.

Theorem 7.4. *$\mathscr{D}'(\Omega)$ is complete.*

Since $u_j(\varphi)$ is a Cauchy sequence in $\mathbb{C}$, the following limit exist

$$u(\varphi) = \lim_{j \to \infty} u_j(\varphi),$$

and defines a linear functional on $\mathscr{D}(\Omega)$. The difficulty is to show that u is a distribution, i.e. that u satisfies the norm inequality (2.1), or the equivalent formulation in Theorem 2.4. This is a consequence of the Banach-Steinhaus theorem.

Let K be a compact set in Ω. We shall study the space $X = X_K = \{\varphi \in C^\infty(\Omega); \operatorname{supp} \varphi \subset K\}$. We introduce a metric on X by

$$d(\varphi_1, \varphi_2) = \sum_k 2^{-k} \frac{\|\varphi_1 - \varphi_2\|_k}{1 + \|\varphi_1 - \varphi_2\|_k},$$

where $\|\varphi\|_k = \sum_{|\alpha|\le k} \sup_K |\partial^\alpha \varphi|$ and put $\|\varphi\| = d(\varphi, 0)$.

Observe that if $\epsilon > 0$, and we take $N = N_\epsilon$ so that $\sum_{N+1}^\infty 2^{-k} < \frac{\epsilon}{2}$, then

$$\|\varphi\| \le \sum_{k=1}^N 2^{-k}\|\varphi\|_k + \frac{\epsilon}{2} \le \sum_{k=1}^N 2^{-k}\|\varphi\|_N + \frac{\epsilon}{2} \le \|\varphi\|_N + \frac{\epsilon}{2} < \epsilon \ ,$$

if $\|\varphi\|_N < \frac{\epsilon}{2}$.

Exercise 7.2. Show that d is a metric on X.

Exercise 7.3. Show that X is complete.

Exercise 7.4. Show that $\varphi_j \to 0$ in $\mathscr{D}(K)$ if and only if $\|\varphi_j\| \to 0$.

Exercise 7.5. Show that if $\|\varphi_j\| \to 0$ in $\mathscr{D}(K)$, there are positive numbers c_j with $c_j \to \infty$ but $\|c_j\varphi_j\| \to 0$.

The Banach-Steinhaus theorem

Let Λ_α be a family of linear functionals on X with $|\Lambda_\alpha \varphi| \le C_\alpha \|\varphi\|$. Then, either

1) there are $r > 0$ and $C < \infty$ with

$$\sup_\alpha |\Lambda_\alpha \varphi| \le C$$

for all $\varphi \in X$ with $\|\varphi\| \le r$,

or

2) $\sup_\alpha |\Lambda_\alpha \varphi| = \infty$ *for some $\varphi \in X$.*

Now we can complete the proof of Theorem 4. Take φ with support in K. Since $u_j(\varphi)$ converges, 2) can not hold. Thus 1) holds, ie.

$$|u(\varphi)| \le \sup_j |u_j(\varphi)| \le C \text{ if } \|\varphi\| \le r \ .$$

Hence if $\varphi_k \to 0$ in $\mathscr{D}(K)$, $k \to \infty$, Exercise 7.5 implies that $|u(c_k\varphi_k)| \le C$ if k is large enough. Thus $|u(\varphi_k)| \le \frac{C}{c_k} \to 0$, $k \to \infty$, ie. $u \in \mathscr{D}'$.

□

The Banach-Steinhaus theorem is a consequence of

Baire's theorem

Assume that X is a complete metric space. Let $V_1, V_2, \dots$ be open dense sets in X. Then $\cap_i V_i$ is non-empty.

Proof. Let $B_r(\phi) = \{\varphi \in X; d(\varphi, \phi) < r\}$. Since V_i are open and dense vi can successivly choose ϕ_i and r_i with $r_i < \frac{1}{i}$ such that $\overline{B_{r_1}(\phi_1)} \subset V_1$ and $\overline{B_{r_i}(\phi_i)} \subset V_i \cap B_{r_{i-1}}(\phi_{i-1})$, $i = 1, 2, 3, \dots$.

If $i, j \geq n$, then $\phi_i, \phi_j \in B_{r_n}(\phi_n)$, and therefore $d(\phi_1, \phi_j) < \frac{2}{n}$. Thus ϕ_n is a Cauchy sequence, and $\phi_n \to \phi_0$ for some $\phi_0 \in X$. But $\phi_i \in B_{r_n}(\phi_n)$ if $i \geq n$. Hence $\phi_0 \in \overline{B_{r_n}(\phi_n)} \subset V_n$for all n and $\phi_0 \in \cap V_i$. □

Proof of the Banach-Steinhaus theorem. Let $\phi(\varphi) = \sup |\Lambda_\alpha \varphi|$. ϕ is lower semi-continuous, and hence $V_n = \{\varphi; \phi(\varphi) > n\}$ is open. If some V_N isn't dense, then there are φ_0, r with $B_r(\varphi_0) \subset V_N^c$ ie.

$$\{\varphi; \|\varphi - \varphi_0\| < r\} \subset V_N^c\ .$$

Thus if $\|\varphi\| < r$, then $|\Lambda_\alpha(\varphi_0 + \varphi)| \leq N$. This implies
$|\Lambda_\alpha \varphi| \leq |\Lambda_\alpha(\varphi + \varphi_0)| + |\Lambda_\alpha \varphi_0| \leq 2N = C$ if $\|\varphi\| < r$.

On the other hand if all V_n are dense, then there are $\varphi \in \cap V_n$, ie. $\phi(\varphi) = \infty$ or $\sup_\alpha |\Lambda_\alpha \varphi| = \infty$. □

Theorem 7.5. *Assume that $u_j \to u_0$ in $\mathscr{D}'(\Omega)$ and that $u_j \geq 0$. Then u_j converges weakly to a positive measure u_0.*

Proof. Since u_0 is the limit of positive distributions, u_0 is a positve distribution. By Theorem 2.7, u_0 is a positive measure. If $\chi \in C_0^\infty$ is equal to 1 on K, the proof of Theorem 2.7 gave the estimate

$$|u_j(\varphi)| \leq 2u_j(\chi)\|\varphi\|_\infty,$$

when $\varphi \in C_0^\infty$ is supported in K.

Since $u_j(\chi) \to u_0(\chi)$, we have $\sup_j |u_j(\chi)| \leq C$, and we obtan

$$|u_j(\varphi)| \leq C\|\varphi\|_\infty,\ \varphi \in C_0^\infty\ , j = 0, 1, 2, \dots$$

By taking limits, compare Theorem 2.5, this also holds when $\varphi \in C_0$.

Now let $\varphi \in C_0$. We have to prove that $u_j(\varphi) \to u_0(\varphi)$, $j \to \infty$. Take $\varphi_n \in C_0^\infty$ whit $\varphi_n \to \varphi$ uniformly. Then

$$\begin{aligned}|u_j(\varphi) - u_0(\varphi)| &\le |u_j(\varphi) - u_j(\varphi_n)| + |u_j(\varphi_n) - u_0(\varphi_n)| + |u_0(\varphi_n) - u_0(\varphi)| \\ &= |u_j(\varphi - \varphi_n)| + |u_j(\varphi_n) - u_0(\varphi_n)| + |u_0(\varphi_n - \varphi)| \\ &\le 2C\|\varphi - \varphi_n\| + |u_j(\varphi_n) - u_0(\varphi_n)| \ .\end{aligned}$$

Hence

$$\overline{\lim_{j\to\infty}} |u_j(\varphi) - u_0(\varphi)| \le 2C\|\varphi - \varphi_n\|_\infty < \epsilon$$

if n is large enough.

□

Exercise 7.6. Assume that f is analytic in $\Omega = I \times (0, \delta) \subset \mathbb{C}$, where I is an open interval. Show that if $|f(z)| \le C|\mathrm{Imz}|^{-\mathrm{N}}$, then $f(x + i0) = \lim_{y\to 0} f(x + iy)$ exists in the distribution sense and $f(x + i0) \in \mathscr{D}'_{N+1}(I)$.

Exercise 7.7. Compute

a) $\frac{1}{x+i0} + \frac{1}{x-i0}$

and

b) $\frac{1}{x+i0} - \frac{1}{x+i0}$.

Exercise 7.8. H 2.5

Exercise 7.9. H 2.6

Exercise 7.10. H 2.7

Exercise 7.11. H 2.9

Exercise 7.12. H 2.16

Chapter 8

Convolution of distributions

If $u \in L^1_{\text{loc}}$ and $\varphi \in C_0^\infty$, then $u * \varphi(x) = \int u(y)\varphi(x-y)dy$. This motivates the following

Definition 8.1. If $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\varphi \in \mathscr{D}(\mathbb{R}^n)$, then

$$u * \varphi(x) = \langle u_y, \varphi(x-y)\rangle.$$

$\square$

The notation $\langle u_y, \varphi(x-y)\rangle$ means that the distribution u acts on the test function $y \mapsto \varphi(x-y)$. Sometimes we also write $\langle u, \varphi(x-\cdot)\rangle$.

Remark 8.2. This definition can also be used in the case where $u \in \mathscr{E}'(\mathbb{R}^n)$, $\varphi \in C^\infty(\mathbb{R}^n)$. $\square$

Theorem 8.1. *If $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\varphi \in \mathscr{D}(\mathbb{R}^n)$, then*

a) $u * \varphi \in C^\infty(\mathbb{R}^n)$

b) $\operatorname{supp}(u * \varphi) \subset \operatorname{supp} u + \operatorname{supp} \varphi$

c) $\partial^\alpha(u * \varphi) = u * \partial^\alpha \varphi = (\partial^\alpha u) * \varphi$

Proof. We first show that $u * \varphi$ is continuous. Let $x \to x_0$. If $|x - x_0| \leq 1$, then $y \mapsto \varphi(x-y)$ has support in a fixed compact set. Furthermore $\partial_y^\alpha(\varphi(x-y) - \varphi(x_0-y)) \to 0$, $x \to x_0$, uniformly. Hence $\varphi(x-y) \to \varphi(x_0-y)$, in $\mathscr{D}$ when $x \to x_0$, and we get $u * \varphi(x) = \langle u_y, \varphi(x-y)\rangle \to \langle u_y, \varphi(x_0-y)\rangle = u * \varphi(x_0), x \to x_0$.

Since $u * \varphi$ is continuous, to prove b) it is enough to show that if $x \notin \operatorname{supp} u + \operatorname{supp} \varphi$, then $u * \varphi(x) = 0$. But if $x \notin \operatorname{supp} u + \operatorname{supp} \varphi$, there are no $y \in \operatorname{supp} u$ with $x - y \in \operatorname{supp} \varphi$. So there is no y with $y \in \operatorname{supp} u$ and $y \in \operatorname{supp} \varphi(x-\cdot)$. Hence $\operatorname{supp} u \cap \operatorname{supp} \varphi(x-\cdot) = \emptyset$ och $u * \varphi(x) = 0$.

The proof of the second equality in c) is simple.

$$\partial^\alpha u * \varphi(x) = \langle \partial^\alpha u_y, \varphi(x-y)\rangle = (-1)^{|\alpha|}\langle u_y, \partial_y^\alpha \varphi(x-y)\rangle =$$
$$= \langle u_y, \varphi^{(\alpha)}(x-y)\rangle = u * (\partial^\alpha \varphi)(x).$$

The first equality follows by induction if we can prove it in the special case $\alpha = (1, 0, \ldots, 0)$. Thus it is enough to show that

$$\lim_{h\to 0} \frac{1}{h}(u * \varphi(x + he_1) - u * \varphi(x)) = u * \partial_1 \varphi(x).$$

Let $\phi_{x,h}(y) = \frac{1}{h}(\varphi(x+he_1-y)-\varphi(x-y))$. Then $\frac{1}{h}(u*\varphi(x+he_1)-u*\varphi(x)) = u(\phi_{x,h})$. But $\phi_{x,h}(y) \to \frac{\partial \varphi}{\partial x_1}(x-y)$ i $\mathscr{D}$. Hence $\partial^\alpha(u*\varphi)(x) = \lim_{h\to 0} u(\phi_{x,h}) = u_y(\frac{\partial\varphi}{\partial x_1}(x-y)) = u * \partial_1\varphi(x)$.

Since a) follows from c) the theorem is proved. □

Exercise 8.1. Show that $\phi_{x,h}(y) \to \frac{\partial\varphi}{\partial x_1}(x-y)$ i $\mathscr{D}$.

Exercise 8.2. Show that the convolution of functions is associative.

Theorem 8.2. *If $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\varphi, \psi \in \mathscr{D}(\mathbb{R}^n)$, then $(u*\varphi)*\psi = u*(\varphi*\psi)$.*

Remark 8.3. If $u \in \mathscr{E}'(\mathbb{R}^n)$, it is enough that one of φ, ψ has compact support. □

Proof. We have

$$u * (\varphi * \psi)(x) = \langle u_y, \varphi * \psi(x-y)\rangle = \langle u_y, \int_{\mathbb{R}^n} \varphi(x-y-t)\psi(t)dt\rangle \stackrel{?}{=}$$
$$\stackrel{?}{=} \int_{\mathbb{R}^n} \langle u_y, \varphi(x-y-t)\rangle\psi(t)dt = \int_{\mathbb{R}^n} u * \varphi(x-t)\psi(t)dt$$
$$= (u * \varphi) * \psi(x).$$

To prove that $\stackrel{?}{=}$ holds, we approximate the integral with its Riemann sum. By Lemma 4 below the Riemann sum converges to the convolution in $\mathscr{D}$ and thus $\stackrel{?}{=}$ holds. □

Lemma 8.4. *If $\varphi \in C_0^j(\mathbb{R}^n)$ and $\psi \in C_0(\mathbb{R}^n)$, then*

$$\sum_{k\in\mathbb{Z}^n} \varphi(x-kh)\psi(kh)h^n \longrightarrow \varphi * \psi(x) \ \ i\ C_0^j,$$

as $h \to 0$.

Proof of Lemma 4. The sum is supported in $\operatorname{supp}\varphi + \operatorname{supp}\psi$. The function $(x, y) \mapsto \varphi(x-y)\psi(y)$ is uniformly continuous. Hence the Riemann sum converges uniformly to $\varphi * \psi(x)$. Since $\partial^\alpha(\varphi * \psi) = \partial^\alpha\varphi * \psi$ om $|\alpha| \le j$, this also holds for the derivatives. □

Theorem 8.5 (Regularisation of distributions.)**.** *Let $u \in \mathscr{D}'(\mathbb{R}^n)$ and φ_δ be an approximate identity. Then $u * \varphi_\delta \to u$ in $\mathscr{D}'(\mathbb{R}^n), \delta \to 0$.*

Proof. Define $\check{\psi}$ by $\check{\psi}(x) = \psi(-x)$. Then $u(\psi) = u * \check{\psi}(0)$. By Theorem 2, this implies

$$u_\delta(\psi) = u * \varphi_\delta(\psi) = (u * \varphi_\delta) * \check{\psi}(0) = \\ = u * (\varphi_\delta * \check{\psi})(0) \ .$$

But, since φ_δ is an approximate identity, $\varphi_\delta * \check{\psi} \to \check{\psi}$ in $\mathscr{D}(\mathbb{R}^n)$, $\delta \to 0$. Hence

$$\lim_{\delta\to 0} u_\delta(\psi) = \lim_{\delta\to 0} u * (\varphi_\delta * \check{\psi})(0) = u * \check{\psi}(0) = u(\psi).$$

□

Exercise 8.3. Let $u \in \mathscr{D}'(\Omega)$. Show that there are C_0^∞ functions u_n with $u_n \to u$ in $\mathscr{D}'(\Omega)$, $n \to \infty$.

Example 8.6. An alternative proof that u is constant if $u' = 0$.

Let $u_\delta = u * \varphi_\delta \in C^\infty$. Then $u'_\delta = u' * \varphi_\delta = 0 * \varphi_\delta = 0$. Hence $u_\delta = C_\delta$. But $u_\delta \to u$ in $\mathscr{D}'$, and $C_\delta \to C$ for some constant C and $u = C$. □

Exercise 8.4. Let $u \in \mathscr{D}'(\mathbb{R})$. Show that

a) If $u' \ge 0$, then u is an increasing function.

b) If $u'' \ge 0$, then u is a convex function.

Example 8.7. Harmonic functions.

If $u \in C^2(\mathbb{R}^n)$ satisfies $\Delta u = 0$, we say that u is a harmonic function. Harmonic functions satisfies the mean value property.

$$u(x) = \frac{1}{|S_r(x)|}\int_{S_r(x)} u(y)d\sigma(y).$$

In $\mathbb{R}^2$, this follows from the Cauchy integral formula since a harmonic function locally is the real part of a holomorphic function. The general case follows from

Exercise 8.5. Prove the mean value property.
Hint. We may assume that $x = 0$. First apply Green's identity to the functions u and 1 on $B_r = \{|x| \le r\}$, and then on u and E (E is the fundamental solution of Δ) on $\Omega_\epsilon = \{\epsilon \le |x| \le 1\}$. Let $\epsilon \to 0$.

A different proof is given in Section 17.2.

Theorem 8.8 (Weyl's lemma). *If $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\Delta u = 0$, then $u \in C^\infty$ and $\Delta u = 0$ classically.*

Proof. Let φ_δ be an approximate identity, $\varphi(x) = \varphi(|x|)$, $\varphi \ge 0$ and $\int \varphi = 1$. Put $u_\delta = u * \varphi_\delta$. Then $u_\delta \in C^\infty$ and $\Delta u_\delta = (\Delta u) * \varphi_\delta = 0 * \varphi_\delta = 0$. So u_δ satisfies the mean value property. Hence

$$\begin{aligned} u_\delta * \varphi(x) &= \int_0^\infty r^{n-1}\varphi(r)dr \int_{S^{n-1}} u_\delta(x - r\omega)d\sigma(\omega) \\ &= \omega_n u_\delta(x) \int_0^\infty r^{n-1}\varphi(r)dr = u_\delta(x) \int_{\mathbb{R}^n} \varphi(y)dy = u_\delta(x). \end{aligned}$$

Thus $u_\delta = u_\delta * \varphi$. Now let $\delta \to 0$. We get $u = u * \varphi \in C^\infty$ and $\Delta u = \Delta u * \varphi = 0 * \varphi = 0$. □

Next we will define the convolution of two distributions. We want to do it in such a way that the associativity is preserved. To be able to that we assume that at least one of the distributions has compact support.

Definition 8.9. Assume that $u, v \in \mathscr{D}'(\mathbb{R}^n)$, and at least one of them has compact support. Then $u * v$ is the (uniquely determined) distribution that satisfies

$$(u * v) * \varphi = u * (v * \varphi), \quad \varphi \in \mathscr{D}(\mathbb{R}^n).$$

□

Is this a definition?

We first observe that $u * (v * \varphi)$ is well-defined. If v has compact support, then $v * \varphi \in \mathscr{D}$ and $u * (v * \varphi)$ is well-defined by Definition 1.1. On the other hand, if u has compact support, then $v * \varphi \in C^\infty$ and $u * (v * \varphi)$ is well-defined by Remark 1.2.

That there is at most one $U = u * v$ is also clear. Namely, if there were two such distribution U and $\widetilde{U}$, then $U * \varphi = u * (v * \varphi) = \widetilde{U} * \varphi$ and $U(\varphi) = U * \check{\varphi}(0) = \widetilde{U} * \check{\varphi}(0) = \widetilde{U}(\varphi)$.

To show the existence we will study the map $\varphi \mapsto u * \varphi$.

Proposition 8.10. *Let $T\varphi = u * \varphi$. Then we have*

a) If $u \in \mathscr{D}'(\mathbb{R}^n)$, then T is a continuous linear map $\mathscr{D}(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$.

b) If $u \in \mathscr{E}'(\mathbb{R}^n)$, then T is a continuous linear map $\mathscr{D}(\mathbb{R}^n) \to \mathscr{D}(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^n) \to C^\infty(\mathbb{R}^n)$.

Proof. We prove a) and leave b) as an exercise. Thus we assume that $\varphi_j \to 0$ i $\mathscr{D}(\mathbb{R}^n)$ and shall prove that $\partial^\alpha(u*\varphi_j) \to 0$ uniformly on compact sets. Since $\partial^\alpha \varphi_j \to 0$ in $\mathscr{D}(\mathbb{R}^n)$ if $\varphi_j \to 0$ in $\mathscr{D}(\mathbb{R}^n)$, and $\partial^\alpha(u * \varphi_j) = u * \partial^\alpha \varphi_j$, we may assume that $\alpha = 0$. If x is contained in a compact set and if all φ_j are supported in another compact set, then also $y \mapsto \varphi_j(x-y)$ is supported in a fix compact set. Thus

$$|u * \varphi_j(x)| = |u(\varphi_j(x - \cdot)| \leq C \sum_{|\alpha| \leq k} \|\partial^\alpha \varphi_j(x - \cdot)\|_\infty \longrightarrow 0,\ j \to \infty.$$

□

Exercise 8.6. Prove Proposition 10 b).

Let τ_h be the translation operator, $\tau_h \varphi(x) = \varphi(x-h)$. Then we have

Proposition 8.11. *Convolution and translation commutes.*

Proof.

$$\begin{aligned} u * \tau_h \varphi(x) &= \langle u_y, \tau_h \varphi(x-y) \rangle = \langle u_y, \varphi(x-h-y) \rangle \\ &= u * \varphi(x-h) = \tau_h(u * \varphi)(x). \end{aligned}$$

□

An important converse of this is

Theorem 8.12. *Assume that T is a continuous linear map from $\mathscr{D}(\mathbb{R}^n)$ into $C^\infty(\mathbb{R}^n)$ that commutes with translations. Then there is a distribution $u \in \mathscr{D}'(\mathbb{R}^n)$ with*

$$T\varphi = u * \varphi, \quad \varphi \in \mathscr{D}(\mathbb{R}^n).$$

Proof. If $T\varphi = u * \varphi$, then in particular $u(\varphi) = u * \check{\varphi}\,(0) = T\,\check{\varphi}\,(0)$. We therefore define u by

$$u(\varphi) = T\,\check{\varphi}\,(0).$$

The continuity assumption implies that u is a distribution. Furthermore we have

$$\begin{aligned} u * \varphi(h) &= \langle u, \varphi(h-x) \rangle = \langle u, \tau_h\,\check{\varphi} \rangle = T((\tau_h\,\check{\varphi})^\vee)(0) \\ &= T(\tau_{-h}\varphi)(0) = \tau_{-h}T(\varphi)(0) = T\varphi(h)\ . \end{aligned}$$

□

The above results implies that Definition 9 is a definition. Proposition 10 shows that $\varphi \mapsto u*(v*\varphi)$ satisfies the conditions in Theorem 12 and, $u*v$ is this distribution. $\square$

Remark 8.13.

a) If $v \in \mathscr{D}(\mathbb{R}^n)$, Definitions 1 and 2 coincides.

b) If both u and v have compact support, then $(u*v)*\varphi = u*(v*\varphi)$ for all $\varphi \in C^\infty(\mathbb{R}^n)$.

$\square$

Example 8.14. $u*\delta = u$ since $(u*\delta)*\varphi = u*(\delta*\varphi) = u*\varphi$. $\square$

Theorem 8.15.

a) $u*v = v*u$

b) $supp\,(u*v) \subset supp\,u + supp\,v$

c) $u*(v*w) = (u*v)*w$
if at least two of the distributions have compact support.

Proof. a) To show that two distributions U and V coincides, it is enough to show that $U*(\varphi*\psi) = V*(\varphi*\psi)$, if $\varphi, \psi \in \mathscr{D}(\mathbb{R}^n)$. Namely, in that case, $(U*\varphi)*\psi = U*(\varphi*\psi) = V*(\varphi*\psi) = (V*\varphi)*\psi$, according to Theorem 2. This implies $U*\varphi = V*\varphi$, and $U = V$.

Now

$$(u*v)*(\varphi*\psi) = u*(v*(\varphi*\psi)) = u*((v*\varphi)*\psi)$$
$$= u*(\psi*(v*\varphi)) = (u*\psi)*(v*\varphi).$$

If v has compact support, the last equality follows by Theorem 8.2. If v does not have compact support it follows from the next exercise.

We also have

$$(v*u)*(\varphi*\psi) = (v*u)*(\psi*\varphi) = (v*\varphi)*(u*\psi) = (u*\psi)*(v*\varphi),$$

and a) is proved.

b) By the commutativity we may assume that v has compact support. Definie $\check{v}$ by $\langle \check{v}, \varphi\rangle = \langle v, \check{\varphi}\rangle$. If $x \in \operatorname{supp}(u*v)$, there is to every $\epsilon > 0$ a $\varphi \in \mathscr{D}(\mathbb{R}^n)$, $\operatorname{supp}\varphi \subset \{y; |x-y| < \epsilon\} = O_\epsilon$, with $0 \neq u*v(\varphi) = u*v*\check{\varphi}\,(0)$ $= u((v*\check{\varphi})^\vee) = u(\check{v}*\varphi)$. So $E = \operatorname{supp} u \cap \operatorname{supp}(\check{v}*\varphi) \neq \emptyset$. Let $y \in E$. Then

$y \in \operatorname{supp} u$ och $y \in \operatorname{supp} \overset{\vee}{v} * \varphi$, or $y = -z + x + \delta$, where $z \in \operatorname{supp} v$ and $|\delta| < \epsilon$. Thus $x = y + z - \delta \in \operatorname{supp} u + \operatorname{supp} v + O_\epsilon$. Now let $\epsilon \to 0$.

c) Assume first that w has compact support. Then $w * \varphi \in \mathscr{D}$, and we get

$$((u * v) * w) * \varphi = (u * v) * (w * \varphi) = u * (v * (w * \varphi)).$$

But also,

$$(u * (v * w)) * \varphi = u * ((v * w) * \varphi) = u * (v * (w * \varphi))$$

and hence $u * (v * w) = (u * v) * w$.

If w does not have compact support, both u and v have, and a) implies

$$\begin{aligned} u * (v * w) &= (v * w) * u = v * (w * u) = (w * u) * v \\ &= w * (u * v) = (u * v) * w. \end{aligned}$$

□

Exercise 8.7. Show that $u * (\psi * \varphi) = (u * \psi) * \varphi$ if $u \in \mathscr{E}'$, $\psi \in \mathscr{D}$ and $\varphi \in C^\infty$.

Theorem 8.16. *$\partial^\alpha(u * v) = \partial^\alpha u * v = u * \partial^\alpha v$ if at least one of the distributions have compact support.*

Proof. If $u \in \mathscr{D}'(\mathbb{R}^n)$, we have $\partial^\alpha u = \partial^\alpha \delta * u$, since

$$\partial^\alpha u * \varphi = u * \partial^\alpha \varphi = u * (\delta * \partial^\alpha \varphi) = u * (\partial^\alpha \delta * \varphi) = (u * \partial^\alpha \delta) * \varphi.$$

Using this we get

$$\partial^\alpha(u * v) = \partial^\alpha \delta * (u * v) = (\partial^\alpha \delta * u) * v = \partial^\alpha u * v.$$

The second equality follows from Theorem 15 a). □

Theorem 8.17. *Assume that $u \in \mathscr{D}'_k$ and $v \in C_0^k$ (or $u \in \mathscr{E}'_k$, $v \in C^k$). Then $u * v$ is the continuous function $x \mapsto \langle u_y, v(x - y)\rangle$.*

Proof. If $x \to x_0$, then $v(x - \cdot) \to v(x_0 - \cdot)$ i C_0^k. But u is continuous on C_0^k, and we get $\langle u_y, v(x - y)\rangle \to \langle u_y, v(x_0 - y)\rangle$. Thus $h(x) = \langle u_y, v(x - y)\rangle$ is a continuous function.

According to Definition 9, $(u * v) * \psi = u * (v * \psi)$. As in the proof of Theorem 2 one can show that $h * \psi = u * (v * \psi)$. Hence $h = u * v$.

□

Exercise 8.8. Let $u, v \in \mathscr{D}'(\mathbb{R})$ with support in $\{x \geq 0\}$. Define $u * v$.

Exercise 8.9. H 4.2.1

Exercise 8.10. H 4.2.2

Exercise 8.11. H 4.2.3

Exercise 8.12. H 4.2.4

Chapter 9

Fundamental solutions

Let
$$P = \sum_{|\alpha \leq N|} a_\alpha \partial^\alpha$$
be a differential operator with constant coefficients and E a fundamental solution to P, ie. $E \in \mathscr{D}'(\mathbb{R}^n)$ and $PE = \delta$. Then
$$P(E * f) = f, \quad f \in \mathscr{E}'(\mathbb{R}^n), \tag{9.1}$$
and
$$E * Pu = u, \quad u \in \mathscr{E}'(\mathbb{R}^n). \tag{9.2}$$
Thus E is both a left and a right invers to P on $\mathscr{E}'$. (But on different domains, so it does not imply that P is bijective.) So (1) gives a solution $u = E * f$ of the equation $Pu = f$ if f has compact support. (2) can be used to study regularity of solutions of $Pu = f$.

Remark 9.1. In Chapter 14 we will show that every differential operator with constant coefficients has a fundamental solution. □

In Chapter 5, we obtained fundamental solutions to the Laplace and heat equations. Another example is that
$$E_k(x) = \begin{cases} (x_1 \dots x_n)^k/(k!)^n, & \text{all } x_i > 0 \\ 0 & \text{otherwise,} \end{cases}$$
is a fundamental solution to $P_{k+1} = \partial_1^{k+1} \dots \partial_n^{k+1}$. Using this we can prove

Theorem 9.2. *If $u \in \mathscr{E}'_m(\mathbb{R}^n)$, there is a continuous function f with*
$$\partial_1^{m+2} \dots \partial_n^{m+2} f = u.$$

Proof. E_{m+1} is a fundamental solution to P_{m+2}. Thus $f = E_{m+1} * u$ satisfies $P_{m+2} f = u$. By Theorem 8.17, f is continuous. $\square$

A corollary of Theorem 1 is the following representation theorem for distributions.

Theorem 9.3. *If $u \in \mathscr{D}'(\Omega)$, there are functions $f_\alpha \in C(\Omega)$ with*

$$u = \sum \partial^\alpha f_\alpha$$

in $\mathscr{D}'$. The sum is locally finite, and if u has finite order the sum is finite.

Proof. Choose a partition of unity $\psi_i \in C_0^\infty$ and $\chi_i \in C_0^\infty$ with $\chi_i = 1$ on $\operatorname{supp} \psi_i$. This can be done in such a way that $\Sigma \chi_i$ is locally finite. We get

$$u(\varphi) = \sum_i \psi_i u(\varphi) = \sum_i \chi_i u(\psi_i \varphi).$$

The distribution $\chi_i u$ has compact support, and hence finite order. Theorem 1 implies that $\chi_i u = \partial^{\alpha_i} f_i$, $f_i \in C$. Hence

$$u(\varphi) = \sum_i \partial^{\alpha_i} f_i(\psi_i \varphi) = \sum_i (-1)^{|\alpha_i|} \int_{\mathbb{R}^n} f_i \partial^{\alpha_i}(\psi_i \varphi)\, dx \ .$$

If we compute $\partial^{\alpha_i}(\psi_i \varphi)$, we get

$$u(\varphi) = \sum_i \sum_\alpha (-1)^{|\alpha|} \int_{\mathbb{R}^n} f_{i,\alpha} \partial^\alpha \varphi\, dx = \sum_\alpha \sum_i \partial^\alpha f_{i,\alpha}(\varphi) \ .$$

Finally, we let $f_\alpha = \sum_i f_{i,\alpha}$. $\square$

To study the regularity of solutions of the equation $Pu = f$, we want to study the set where u is not C^∞.

Definition 9.4. The singular support of a distribution $u \in \mathscr{D}'(\Omega)$ is denoted $\operatorname{sing\,supp} u$, and consists of those points in Ω that have no neighborhood where u is C^∞. $\square$

$\operatorname{sing\,supp} u$ is the smallest closed set such that u is C^∞ in its complement. It is clear that $\operatorname{sing\,supp} u \subset \operatorname{supp} u$.

Theorem 9.5. *If $u, v \in \mathscr{D}'(\mathbb{R}^n)$, and at least one of them have compact support, then*

$$\operatorname{sing\,supp}(u * v) \subset \operatorname{sing\,supp} u + \operatorname{sing\,supp} v. \tag{9.3}$$

Proof. Put $u_1 = u$ and $u_2 = v$. Let us first assume that both distributions have compact support. Let $K_i = $ sing supp u_i, and Ω_i a neighborhood of K_i and take $\psi_i \in C_0^\infty(\Omega_i)$ with $\psi_i = 1$ on K_i. Then

$$\begin{aligned} u_1 * u_2 = &(\psi_1 u_1 + (1-\psi_1)u_1) * (\psi_2 u_2 + (1-\psi_2)u_2) \\ =&\psi_1 u_1 * \psi_2 u_2 + \psi_1 u_1 * (1-\psi_2)u_2 \\ &+(1-\psi_1)u_1 * \psi_2 u_2 + (1-\psi_1)u_1 * (1-\psi_2)u_2. \end{aligned}$$

Since $(1-\psi_i)u_i \in C_0^\infty$, Theorem 8.1 implies that the last three terms are C^∞. Thus

$$\begin{aligned} &\operatorname{sing\,supp}(u_1 * u_2) = \operatorname{sing\,supp}(\psi_1 u_1 * \psi_2 u_2) \\ &\subset \operatorname{supp}(\psi_1 u_1 * \psi_2 u_2) \subset \operatorname{supp}\psi_1 + \operatorname{supp}\psi_2 \subset \Omega_1 + \Omega_2. \end{aligned}$$

If we let $\Omega_i \downarrow K_i$, we obtain (9.3).

If only one of the distributions have compact support, we can by Theorem 8.15 asssume that $v \in \mathscr{E}'$.

To show that $\operatorname{sing\,supp} u * v \subset \operatorname{sing\,supp} u + \operatorname{sing\,supp} v$, it is enough to show that $\operatorname{sing\,supp} u * v \cap B_1(x) \subset (\operatorname{sing\,supp} u + \operatorname{sing\,supp} v) \cap B_1(x)$ for each $x \in \mathbb{R}^n$.

Take $R \geq 1$ so large that $\operatorname{supp} v \subset B_R(0)$ and $|x| \leq R$, and choose $\chi \in C_0^\infty(B_{6R}(0))$ with $\chi = 1$ on $B_{5R}(0)$. Put $u_1 = \chi u$ and $u_2 = (1-\chi)u$. Thus $u = u_1 + u_2$ where u_1 has compact support and $\operatorname{supp} u_2 \subset B_{5R}^C(0)$. Then $\operatorname{supp} u_2 * v \subset B_{5R}^C(0) + B_R(0) \subset B_{4R}^C(0)$ and $B_1(x) \subset B_{2R}(0)$. Hence $u_2 * v = 0$ on $B_1(x)$. Since both u_1 and v have compact support, we get

$$\begin{aligned} &\operatorname{sing\,supp}(u * v) \cap B_1(x) = \operatorname{sing\,supp}(u_1 * v) \cap B_1(x) \\ \subset &(\operatorname{sing\,supp} u_1 + \operatorname{sing\,supp} v) \cap B_1(x) = (\operatorname{sing\,supp} u + \operatorname{sing\,supp} v) \cap B_1(x), \end{aligned}$$

and the theorem is proved. □

Theorem 9.6. *If P has a fundamental solution with* $\operatorname{sing\,supp} E = \{0\}$*, then* $\operatorname{sing\,supp} u = \operatorname{sing\,supp} Pu$*, $u \in \mathscr{D}'$.*

Remark 9.7. The converse is also true. Thus if there is a fundamental solution with singular support at the origin, then this is true for all fundamental solutions. □

Proof. $\operatorname{sing\,supp} Pu \subset \operatorname{sing\,supp} u$ allways holds since Pu is C^∞ if u is.

For the other inclusion, we first observe that if u has compact support, then $u = E * Pu$ and by Theorem 5

$$\operatorname{sing\,supp} u \subset \operatorname{sing\,supp} E + \operatorname{sing\,supp} Pu = \operatorname{sing\,supp} Pu.$$

If u is not compactly supported, take $\psi \in C_0^\infty$ with $\psi = 1$ on an open set Ω. Then

$$\operatorname{sing\,supp} \psi u \subset \operatorname{sing\,supp} P(\psi u).$$

But on Ω we have $P(\psi u) = Pu$ and $\psi u = u$, and the result follows. □

A differential operator P is called *hypoelliptisk* if every solution u of $Pu = f$ is C^∞ if f is. Theorem 6 thus implies that P is hypoelliptic if P has a fundamental solution E with $\operatorname{sing\,supp} E = \{0\}$.

The Laplace and the heat operators are hypoelliptic. In Chapter 12, we will show that all elliptic operators are hypoelliptic. The Laplace operator is elliptic but not the heat operator.

Exercise 9.1. Let P be a polynomial of one variable. Show that $P(\frac{d}{dx})$ has a fundamental solution.

Exercise 9.2. H 4.4.3

Exercise 9.3. H 4.4.4

Exercise 9.4. H 4.4.5

Exercise 9.5. H 4.4.6

Exercise 9.6. H 4.4.9

Chapter 10

The Fourier transform

If u is a "nice" periodic function with period T, u can be written as

$$u(x) = \sum_m c_m e^{2\pi i m x/T}. \tag{10.1}$$

Then

$$u(x)e^{-2\pi i \nu x/T} = \sum_\nu c_\nu e^{2\pi i (m-\nu)x/T},$$

and integration over $[-\frac{T}{2}, \frac{T}{2}]$ gives formally that

$$Tc_\nu = \int_{-\frac{T}{2}}^{\frac{T}{2}} u(x)^{-2\pi i \nu x/T} dx$$

eller

$$c_\nu = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} u(x)e^{-2\pi i \nu x/T} dx$$

c_ν are the Fourier coefficients of u. (1) is the inversion theorem. We also have Parseval's identity

$$\sum |c_\nu|^2 = \frac{1}{2\pi} \|u\|_2^2 .$$

How can this be generalised to $\mathbb{R}^n$? Let us first consider the case $n = 1$. Let $u \in C_0^\infty(\mathbb{R})$ and choose T so that $\operatorname{supp} u \subset (-\frac{T}{2}, \frac{T}{2})$. Let u_T be the periodic extension of u,

$$u_T(x) = \sum_{k \in \mathbb{Z}} u(x - kT).$$

Then we have

$$c_T(\nu) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} u(x)e^{-i\frac{2\pi\nu}{T}x}dx.$$

Thus for $|x| < \frac{T}{2}$, (1) implies

$$u(x) = u_T(x) = \sum_\nu c_T(\nu)e^{2\pi i\frac{\nu}{T}x}.$$

Define

$$\widehat{u}(\xi) = \int_{\mathbb{R}} u(x)e^{-i\xi x}dx, \quad \xi \in \mathbb{R}.$$

We observe that $c_T(\nu) = \frac{1}{T}\widehat{u}(\frac{2\pi\nu}{T})$, and we can write

$$u(x) = \sum_\nu \frac{1}{T}\widehat{u}(\frac{2\pi\nu}{T})e^{2\pi i\frac{\nu}{T}x} = \frac{1}{2\pi}\sum_\nu \frac{2\pi}{T}\widehat{u}(\frac{2\pi\nu}{T})e^{i\frac{2\pi\nu}{T}x}\ .$$

This is a Riemann sum of the integral

$$\frac{1}{2\pi}\int_{\mathbb{R}} \widehat{u}(\xi)e^{i\xi x}d\xi.$$

So, if we let $T \to \infty$, we obtain

$$u(x) = \frac{1}{2\pi}\int_{\mathbb{R}} \widehat{u}(\xi)e^{i\xi x}d\xi. \tag{10.2}$$

With some care, the above argument can be used to prove (2) when u is a nice function. We will not do this, but instead prove (2) (and its generalization to $\mathbb{R}^n$) directly. The theory for Fourier series will then be a corollary of the theory of the Fourier transform.

Definition 10.1. Assume that $f \in L^1(\mathbb{R}^n)$. The Fourier transform of f is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-ix\xi}dx,$$

where $x\xi = \sum_1^n x_i\xi_i$. We sometimes write $\mathcal{F}f$ instead of $\widehat{f}$.

□

We will prove the following important properties of the Fourier transform.

I. The inversion formula. If f and $\widehat{f} \in L^1$, then

$$f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{ix\xi}d\xi.$$

II. Parseval's identity. If $f \in L^1 \cap L^2$, then $\widehat{f} \in L^2$ and $\|f\|_2 = \frac{1}{(2\pi)^n}\|\widehat{f}\|_2$.

III. If $f, g \in L^1$, then $(f * g)^\wedge = \widehat{f}\,\widehat{g}$.

IV. $\mathcal{F}(P(D)f)(\xi) = P(\xi)\widehat{f}(\xi)$ where $D_j = -i\partial_j$.

Exercise 10.1. Prove the Riemann-Lebesgue lemma: If $f \in L^1$, then $\widehat{f}$ is continuous and $\widehat{f}(\xi) \to 0$ when $|\xi| \to \infty$.

Exercise 10.2. Prove III and IV.

To solve the constant coefficient differential equation $P(D)u = f$, we can use I and IV. By Fourier transformation, we get $P(\xi)\widehat{u}(\xi) = \widehat{f}(\xi)$. Thus $\widehat{u}(\xi) = \widehat{f}(\xi)/P(\xi)$ and $u = \mathcal{F}^{-1}(\widehat{f}/P)$. To be able to use this method "often", we want to extend the Fourier transform to distributions. As a motivation for the definition, we observe that, by Fubini's theorem we have

Proposition 10.2. *If $f, g \in L^1$, then* $\displaystyle\int_{\mathbb{R}^n} f\widehat{g}\,dx = \int_{\mathbb{R}^n} \widehat{f}g\,dx.$

Exercise 10.3. Prove Proposition 2.

Hence for a L^1 function we have

$$\langle \widehat{f}, \varphi \rangle = \int_{\mathbb{R}^n} \widehat{f}\varphi\,dx = \int_{\mathbb{R}^n} f\widehat{\varphi}\,dx = \langle f, \widehat{\varphi} \rangle.$$

It is therefore natural to define $\widehat{u}$ when $u \in \mathscr{D}'$ by

$$\langle \widehat{u}, \varphi \rangle = \langle u, \widehat{\varphi} \rangle. \tag{10.3}$$

But if $\varphi \in \mathscr{D}, \varphi \not\equiv 0$, then $\widehat{\varphi}$ can not have compact support and hence $\widehat{\varphi} \notin \mathscr{D}$. So we can not define $\langle u, \widehat{\varphi} \rangle$ by (3).

So what to do? Well, we will consider a different class of test functions, that is preserved by the Fourier transform. This is the Schwartz space $\mathscr{S}$, that consists of C^∞ rapidly decreasing functions.

Definition 10.3.

(a) $\varphi \in \mathscr{S}(\mathbb{R}^n)$ if $\varphi \in C^\infty$ and $\sup_{x \in \mathbb{R}^n}(1 + |x|^2)^k |\partial^\alpha \varphi(x)| < \infty$ for alla k and α.

(b) $\varphi_j \to 0$ in $\mathscr{S}(\mathbb{R}^n)$ if

$$\sup_{x \in \mathbb{R}^n}(1 + |x|^2)^k |\partial^\alpha \varphi_j(x)| \to 0$$

for all k and α. □

Definition 10.4.

(a) A *tempered distribution* on $\mathbb{R}^n$ is a linear functional on $\mathscr{S}$, such that $u(\varphi_j) \to 0$ when $\varphi_j \to 0$ in $\mathscr{S}$. We write $u \in \mathscr{S}'$.

(b) A sequence $u_j \in \mathscr{S}'$ converges to $u \in \mathscr{S}'$ if

$$u_j(\varphi) \to u(\varphi),$$

for every testfunction $\varphi \in \mathscr{S}$.

$\square$

To show that (3) works as a definition of the Fourier transform if $u \in \mathscr{S}'$ we need to study the Fourier transform on $\mathscr{S}$. We start with the following

Proposition 10.5. *If $f, g \in \mathscr{S}$, then*

(a) $$\mathcal{F}(x^\alpha f(x)) = i^\alpha \partial^\alpha \widehat{f}$$

(b) $$(\partial^\alpha f)^\wedge(\xi) = (i\xi)^\alpha \widehat{f}(\xi)$$

(c) $$(\tau_h f)^\wedge(\xi) = e^{-i\xi h} \widehat{f}(\xi)$$

(d) $$\mathcal{F}(e^{ixh} f(x)) = \tau_h \widehat{f}$$

(e) $$\widehat{f_a}(\xi) = \widehat{f}(a\xi)$$

(f) $$(f(ax))^\wedge = (\widehat{f})_a$$

(g) $$\int_{\mathbb{R}^n} f\,\widehat{g} = \int_{\mathbb{R}^n} \widehat{f} g$$

(h) $$(f * g)^\wedge = \widehat{f}\,\widehat{g}$$

and

(i) $$\widehat{f} \in \mathscr{S}.$$

Exercise 10.4. Prove Proposition 5.

Theorem 10.6 (The inversion theorem)**.** *If $f \in \mathscr{S}$, then*

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ix\xi} \widehat{f}(\xi) d\xi. \tag{10.4}$$

To prove this we need to find *one* function that satisfies (4). Then (4) follows in general by Proposition 5. We make the choice $G(x) = e^{-|x|^2/2}$.

Lemma 10.7. *$\widehat{G} = (2\pi)^{n/2} G$.*

Proof. By Fubinin's theorem it is enough to consider the case $n = 1$. G satisfies the differential equation

$$G'(x) + xG(x) = 0.$$

If we take the Fourier transform of this equation, Proposition 5 (a) and (b) implies

$$i\xi\widehat{G}(\xi) + i\widehat{G}'(\xi) = 0,$$

or

$$\widehat{G}'(\xi) + \xi\widehat{G}(\xi) = 0.$$

Hence $\widehat{G}(\xi) = CG(\xi)$. If we let $\xi = 0$, we get

$$C = \widehat{G}(0) = \int_{\mathbb{R}} e^{-x^2/2} dx = \sqrt{2\pi}.$$

□

Exercise 10.5.
a) Prove Lemma 7 using the Cauchy theorem.
b) Prove Lemma 7, letting $\xi = \zeta \in \mathbb{C}$, and compute $\widehat{G}(i\eta)$.

Proof of Theorem 6. $\frac{1}{(2\pi)^n}(\widehat{G})_\delta$ is an approximate identity. Proposition 5 f) and g) implies that

$$\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(x)(\widehat{G})_\delta(x)dx = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{f}(\xi)G(\delta\xi)d\xi.$$

Letting $\delta \to 0$, we get

$$f(0) = G(0)\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{f}(\xi)d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{f}(\xi)d\xi.$$

Exercise 10.6. Prove this.

If we apply this to $\tau_{-x}f$, we get

$$f(x) = \tau_{-x}f(0) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} (\tau_{-x}f)^\wedge(\xi)d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{ix\xi}d\xi.$$

□

Remark 10.8. If we only assume that $f \in L^1$, then

$$\frac{1}{(2\pi)^n} f*(\widehat{G})_\delta(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} f(x+y)\widehat{G}_\delta(y)dy = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \widehat{f}(\xi)e^{ix\xi}e^{-\delta^2|\xi|^2/2}d\xi.$$

This implies

$$f(x) = \lim_{\delta\to 0}\frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f}(\xi)e^{ix\xi}e^{-\delta^2|\xi|^2/2}d\xi,$$

with convergence in L^1.

In particular, if $\widehat{f} \in L^1$, then

$$f(x) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{f}(\xi)e^{ix\xi}d\xi \quad \text{a.e.}$$

□

Theorem 10.9 (Plancherel). *If $\phi, \psi \in \mathscr{S}$, then*

$$\int_{\mathbb{R}^n}\phi\bar{\psi}\,dx = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{\phi}\,\overline{\widehat{\psi}}\,d\xi\ .$$

Corollary 10.10 (Parseval). *If $\phi \in \mathscr{S}$, then*

$$\|\phi\|_2 = \frac{1}{(2\pi)^{n/2}}\|\widehat{\phi}\|_2$$

.

Proof. Proposition 2 g) implies

$$\int_{\mathbb{R}^n}\phi\widehat{\psi_0}\,dx = \int_{\mathbb{R}^n}\widehat{\phi}\psi_0\,dx\ .$$

Let $\widehat{\psi_0} = \bar{\psi}$. By the inversion theorem,

$$\begin{aligned}\psi_0(x) &= \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi}\widehat{\psi_0}(\xi)d\xi = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} e^{ix\xi}\bar{\psi}(\xi)d\xi \\ &= \frac{1}{(2\pi)^n}\overline{\int_{\mathbb{R}^n} e^{-ix\xi}\psi(\xi)d\xi} = \frac{1}{(2\pi)^n}\overline{\widehat{\psi}(x)}.\end{aligned}$$

Thus $\displaystyle\int_{\mathbb{R}^n}\phi\bar{\psi} = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n}\widehat{\phi}\overline{\widehat{\psi}}$. The corollary follows by taking $\psi = \phi$. □

Remark 10.11. The Parseval formula also holds if $\phi, \psi \in L^2$. We will prove this in the next chapter. □

To prove that $\widehat{u}$, defined by $\widehat{u}(\varphi) = u(\widehat{\varphi})$, is a tempered distribution we need the following

Lemma 10.12. *$\mathcal{F}:\mathscr{S}\to\mathscr{S}$ continuously, i.e., if $\varphi_j\to 0$ i $\mathscr{S}$, then $\widehat{\varphi}_j\to 0$ i $\mathscr{S}$.*

Proof. Proposition 5 a) and b) implies $\xi^\beta\partial^\alpha\widehat{\varphi}_j(\xi) = c\mathcal{F}\big(\partial^\beta(x^\alpha\varphi_j(x))\big)(\xi)$. Hence

$$\begin{aligned}\sup_\xi|\xi^\beta\partial^\alpha\widehat{\varphi}_j(\xi)| &\le c\sup_\xi\left|\int_{\mathbb{R}^n} e^{-ix\xi}\partial^\beta(x^\alpha\varphi_j(x))dx\right|\\ &\le c\int_{\mathbb{R}^n}|\partial^\beta(x^\alpha\varphi_j(x))|dx\to 0,\end{aligned}$$

as $\varphi_j\to 0$ in $\mathscr{S}$. □

We are now ready to make the following definition.

Definition 10.13. If $u\in\mathscr{S}'$, then $\widehat{u}$ is the tempered distribution given by

$$\widehat{u}(\varphi)=u(\widehat{\varphi}).$$

□

Remark 10.14. We observe that the two definitions of $\widehat{f}$ when $f\in L^1$ coincides. □

Theorem 10.15. *The Fourier transform is a continuous linear bijection from $\mathscr{S}'$ to $\mathscr{S}'$ with $\widehat{\widehat{u}}=(2\pi)^n\overset{\vee}{u}$.*

Proof. We reminde the reader that $\overset{\vee}{u}$ is defined by $\overset{\vee}{u}(\varphi)=u(\overset{\vee}{\varphi})$, and that $u_j\to u$ in $\mathscr{S}'$ means that $u_j(\varphi)\to u(\varphi)$ for all $\varphi\in\mathscr{S}$. The theorem is an easy consequence of the corresponding properties on $\mathscr{S}$:

$$\widehat{\widehat{u}}(\varphi)=\widehat{u}(\widehat{\varphi})=u(\widehat{\widehat{\varphi}})=(2\pi)^n u(\overset{\vee}{\varphi})=(2\pi)^n\,\overset{\vee}{u}(\varphi)$$

and

$$\widehat{u}_j(\varphi)=u_j(\widehat{\varphi})\to u(\widehat{\varphi})=\widehat{u}(\varphi)$$

if $u_j\to u$ i $\mathscr{S}'$. □

Example 10.16. a) A measure μ with $\int_{\mathbb{R}^n}(1+|x|^2)^{-k}d\mu(x)<\infty$ for some k is a tempered distribution.

b) If $f\in L^p, 1\le p\le\infty$, then $f\in\mathscr{S}'$. (Proof. Hölder's inequality.)

c) $\widehat{\delta}=1$ and $\widehat{1}=(2\pi)^n\delta$.

d) e^x is not a tempered distribution. □

Proposition 10.17. *If $u \in \mathscr{S}'$, then*

a) $(x_j u)^\wedge = -D_j \widehat{u}$

b) $(D_j u)^\wedge = \xi_j \widehat{u}$

c) $(\tau_h u)^\wedge(\xi) = \exp(-ih\xi)\widehat{u}(\xi)$

and

d) $\mathcal{F}(\exp(ixh)u) = \tau_h \widehat{u}$.

Proof. It is easy to see that $D_j u, x_j u, \dots$ are tempered distributions. Then the formulas follows from Proposition 5. (Remember that $D_j = -i\partial_j$.) □

Exercise 10.7. Show that $e^x \cos(e^x) \in \mathscr{S}'$.

Exercise 10.8. Show that $u \in \mathscr{S}'$ if and only if

$$|u(\varphi)| \le C \sum_{k+|\alpha|\le N} \sup(1+|x|^2)^k |\partial^\alpha \varphi(x)|$$

for some N.

Exercise 10.9. H 7.1.10

Exercise 10.10. H 7.1.19

Exercise 10.11. H 7.1.20

Exercise 10.12. H 7.1.21

Exercise 10.13. H 7.1.22

Exercise 10.14. H 7.6.1

Chapter 11

The Fourier transform on L^2

According to Exempel 10.16b), $f \in L^2$ has a Fourier transform defined as a tempered distribution. In fact we have the following result.

Theorem 11.1. *If $f \in L^2(\mathbb{R}^2)$, then $\hat{f} \in L^2(\mathbb{R}^n)$ and*

$$\|f\|_2 = \frac{1}{(2\pi)^{n/2}} \|\hat{f}\|_2.$$

Furthermore $\hat{f}$ is given by

$$\hat{f}(\xi) = \lim_{N\to\infty} \int_{|x|\le N} e^{-i\xi x} f(x)dx \ ,$$

with convergence in L^2.

Proof. Take $f_n \in C_0^\infty, f_n \to f$ in L^2. Then f_n is a Cauchy sequence in L^2. By the Plancherel theorem, we have

$$\|\hat{f}_n - \hat{f}_m\|_2 = c\|f_n - f_m\|_2 \to 0, \quad n, m \to \infty.$$

Hence $\hat{f}_n$ is a Cauchy sequence in L^2. By the completeness of L^2, we have $\hat{f}_n \to g$ i L^2 for some $g \in L^2$. This implies that $\hat{f}_n \to g$ in $\mathscr{S}'$. Furthermore $f_n \to f$ in $\mathscr{S}'$, and since the Fourier transform is continuous on $\mathscr{S}'$, we have $\hat{f}_n \to \hat{f}$ in $\mathscr{S}'$. Hence $\hat{f} = g \in L^2$ and we get

$$\|\hat{f}\|_2 = \lim_{n\to\infty} \|\hat{f}_n\|_2 = \frac{1}{(2\pi)^{n/2}} \lim_{n\to\infty} \|f_n\|_2 = \frac{1}{(2\pi)^{n/2}} \|f\|_2 \ ,$$

and the first part is proved.

Put $f_N = f\chi_{\{|x|\le N\}}$. Then $f_N \to f$ in L^2 and $f_N \in L^1$. Hence

$$\hat{f}_N(\xi) = \int_{|x|\le N} e^{-ix\xi} f(x)dx,$$

and by the Plancherel theorem, we obtain

$$\|\hat{f} - \hat{f}_N\|_2 = c\|f - f_N\|_2 \to 0, \quad N \to \infty,$$

and the proof is complete.

□

Chapter 12

The Fourier transform and convolutions

We shall show that under suitable conditions $(u * v)^\wedge = \widehat{u}\,\widehat{v}$.

First we observe that $\mathscr{D} \subset \mathscr{S} \subset C^\infty$. The inclusions are continuous, ie. if $\varphi_j \to 0$ i $\mathscr{D}$, then $\varphi_j \to 0$ in $\mathscr{S}$, and this in turn implies that $\varphi_j \to 0$ in C^∞. Furthermore, $\mathscr{D}$ is dense in $\mathscr{S}$, and $\mathscr{S}$ is dense in C^∞. (Show that!) Hence

$$\mathscr{E}'(\mathbb{R}^n) \subset \mathscr{S}'(\mathbb{R}^n) \subset \mathscr{D}'(\mathbb{R}^n).$$

Definition 12.1. If $u \in \mathscr{S}'$ and $\phi \in \mathscr{S}$, we define the convolution $u * \phi$ by $u * \phi(x) = \langle u_y, \phi(x - y)\rangle$. □

Theorem 12.2. *If $u \in \mathscr{S}'$ and $\phi \in \mathscr{S}$, then*

(a) $u * \phi \in C^\infty$ *och* $\partial^\alpha(u * \phi) = \partial^\alpha u * \phi = u * \partial^\alpha \phi$

(b) $u * \phi$ *is bounded by a polynomial (and hence $u * \phi \in \mathscr{S}'$), and* $(u * \phi)^\wedge = \widehat{\phi}\,\widehat{u}$.

(c) $u * (\phi * \psi) = (u * \phi) * \psi \quad (\psi \in \mathscr{S})$

and

(d) $\widehat{u} * \widehat{\phi} = (2\pi)^n (\phi u)^\wedge$.

Sketch of proof. (a) We assume that $n = 1$. The second equality is proved in the same way as in Theorem 8.1. As in the proof of Theorem 8.1, the first equality folows if we can prove that

$$\frac{\phi(x + h) - \phi(x)}{h} \longrightarrow \phi'(x) \text{ in } \mathscr{S}.$$

To do this is elementary but tedious. The simplest way is (probably) to use the Fourier transform.

(b) By Exercise 10.8, we have

$$\begin{aligned}|u * \phi(x)| &= |\langle u_y, \phi(x-y)\rangle| \\ &\le C \sup_y \sum_{k+|\alpha|\le N} (1+|y|^2)^k |\partial^\alpha \phi(x-y)| \\ &\le C \sup_y \sum_{k+|\alpha|\le N} (1+|x|^2)^k (1+|x-y|^2)^k |\partial^\alpha \phi(x-y)| \\ &\le C(1+|x|^2)^N.\end{aligned}$$

If $\psi \in \mathscr{D}$, we also have

$$(u*\phi)^\wedge(\widehat{\psi}) = (u*\phi)(\widehat{\widehat{\psi}}) = (2\pi)^n (u*\phi)(\overset{\vee}{\psi}) = (2\pi)^n \int_{\mathbb{R}^n} u*\phi(x)\psi(-x)dx$$

$$= (2\pi)^n \int_{-K} \langle u_y, \psi(-x)\phi(x-y)\rangle dx = \text{ Approximate with a Riemann sum } =$$

$$= (2\pi)^n \langle u_y, \int_{\mathbb{R}^n} \psi(-x)\phi(x-y)dx\rangle = (2\pi)^n \langle u_y, \int_{\mathbb{R}^n} \psi(x)\phi(-y-x)dx\rangle$$

$$= (2\pi)^n \langle u_y, (\phi*\psi)^\vee)\rangle = \langle u_y, (\phi*\psi)^{\wedge\wedge}\rangle = \widehat{u}((\phi*\psi)^\wedge) = \widehat{u}(\widehat{\phi}\widehat{\psi}) = \widehat{\phi}\,\widehat{u}(\widehat{\psi}).$$

But $\widehat{\mathscr{D}}$ is dense in $\mathscr{S}$, and (b) follows.

(c) From the proof of (b), we get

$$u*\phi(\overset{\vee}{\psi}) = u((\phi*\psi)^\vee),$$

first for $\psi \in \mathscr{D}$, and by contiuity also for $\psi \in \mathscr{S}$. This can be written

$$(u*\phi)*\psi(0) = u*(\phi*\psi)(0).$$

The general case follows if we replace ψ with $\tau_{-x}\psi$.

(d) By (b), we have $(\widehat{u}*\widehat{\phi})^\wedge = \widehat{\widehat{\phi}}\,\widehat{\widehat{u}} = (2\pi)^{2n}\,\overset{\vee}{\phi}\overset{\vee}{u} = (2\pi)^{2n}(\phi u)^\vee = (2\pi)^n(\phi u)\widehat{\ }\widehat{\ }$. Thus, by the inversion theorem we get $\widehat{u}*\widehat{\phi} = (2\pi)^n(\phi u)\widehat{\ }$.

□

Theorem 12.3. *If $u \in \mathscr{E}'(\mathbb{R}^n)$, then $\widehat{u} \in C^\infty$ and $\widehat{u}(\xi) = u_x(e^{-ix\xi})$.*

Proof. Let $\psi \in C_0^\infty$ be 1 on a neighborhood of supp u. Then $u = \psi u$ and by Theorem 2(d) we get $\widehat{u} = (\psi u)\widehat{\ } = (2\pi)^{-n}\widehat{u}*\widehat{\psi} \in C^\infty$. Thus

$$\begin{aligned}\widehat{u}(\xi) &= (2\pi)^{-n}\widehat{u}*\widehat{\psi}(\xi) = (2\pi)^{-n}\langle \widehat{u}_x, \widehat{\psi}(\xi-x)\rangle = (2\pi)^{-n}\langle \widehat{u}_x, \overset{\vee}{\widehat{\psi}}\,(x-\xi)\rangle \\ &= (2\pi)^{-2n}\langle \widehat{u}_x, \mathcal{F}^3\psi(x-\xi)\rangle = (2\pi)^{-2n}\langle \widehat{u}_x, \tau_\xi \mathcal{F}^3\psi(x)\rangle \\ &= (2\pi)^{-2n}\langle u_x, e^{-ix\xi}\mathcal{F}^4\psi(x)\rangle = u_x(e^{-i\xi x}\psi(x)) = u_x(e^{-i\xi x}).\end{aligned}$$

□

Remark 12.4. In the next chapter we will prove the Paley-Wiener theorem that gives much more precise information of $\widehat{u}$ when $u \in \mathscr{E}'$. □

Example 12.5. Determine the Fourier transform of $\mathrm{pv}\dfrac{1}{x}$. □

Method 1. Let $u = \mathrm{pv}\dfrac{1}{x}$. Then u is the sum of a distribution in $\mathscr{E}'$ and an L^2 function. Thus

$$\widehat{u}(\xi) = \lim_{\substack{\epsilon\to 0\\ N\to\infty}} \int_{\epsilon<|x|<N} e^{-ix\xi}\frac{dx}{x}.$$

If $\xi > 0$, the change of variables $y = x\xi$ implies,

$$\widehat{u}(\xi) = \lim_{\substack{\epsilon\to 0\\ N\to\infty}} \int_{\epsilon<|x|<N} \frac{e^{-iy}}{y}dy = -i\int_{-\infty}^{\infty}\frac{\sin y}{y}dy = -i\pi.$$

When $\xi < 0$, we instead get $\widehat{u}(\xi) = i\pi$. Hence

$$\widehat{u}(\xi) = -\pi \text{ sgn } \xi.$$

Exercise 12.1. Prove that $\int_{-\infty}^{\infty}\frac{\sin y}{y}dy = \pi$.

Method 2. We have $x\mathrm{pv}\dfrac{1}{x} = 1$. This implies $i\widehat{u}' = 2\pi\delta$, $\widehat{u}' = -2\pi i\delta$ and $\widehat{u} = -2\pi i(H + c)$. Since u is odd, so is $\widehat{u}$. Hence $c = -\frac{1}{2}$ och

$$\widehat{u}(\xi) = -i\pi \text{ sgn } \xi.$$

In the last argument we used that if a distribution u is odd, then $\widehat{u}$ is also odd. This is clear if $u \in L^1$ (a simple change of variables).

Definition 12.6. A distribution is even if $\check{u} = u$, and odd if $\check{u} = -u$. □

Proposition 12.7. *If u is an odd tempered distribution, then its Fourier transform is also odd.*

Proof. By Theorem 10.15

$$\check{\widehat{u}} = (2\pi)^{-n}\,\widehat{\widehat{\widehat{u}}} = \left((2\pi)^{-n}\,\widehat{\widehat{u}}\right)^{\wedge} = \widehat{\check{u}} = (u \text{ is odd}) = -\widehat{u}\,.$$

□

In the same way we see that the Fourier transform of an even distribution is even.

Remark 12.8. The map $H\varphi = \text{pv}\frac{1}{x} * \varphi$, is called the the Hilbert transform. The Hilbert transform is an important example of a so called singular integral operator. The Hilbert transform is bounded on L^p, $1 < p < \infty$, and of weak-type $(1,1)$.

When $p = 2$, this follows from Exampel 5 and the Plancherel theorem. □

Next we will study invariance properties of the Fourier transform. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism (i.e. a C^∞ bijection). If u is a function, we have

$$\int_{\mathbb{R}^n} u \circ F(x)\varphi(x)dx = \int_{\mathbb{R}^n} u(y)\frac{\varphi}{|F'|} \circ F^{-1}(y)dy.$$

Therefore, if $u \in \mathscr{D}'$, we define $u \circ F$ by

$$\langle u \circ F, \varphi\rangle = \langle u, \frac{\varphi}{|F'|} \circ F^{-1}\rangle$$

In particular, if $F = \Lambda$ is linear, then

$$\langle u \circ \Lambda, \varphi\rangle = |\det \Lambda|^{-1}\langle u, \varphi \circ \Lambda^{-1}\rangle$$

□

Definition 12.9. A distribution u is radial if $u \circ O = u$ for all orthogonal maps O.

Theorem 12.10. *If u is a radial tempered distribution, then $\widehat{u}$ is radial.*

Proof. First, we observe that if $\varphi \in \mathscr{S}$, then

$$\begin{aligned}\widehat{\varphi} \circ O(\xi) = \widehat{\varphi}(O\xi) &= \int_{\mathbb{R}^n} e^{-ixO\xi}\varphi(x)dx \\ &= \int_{\mathbb{R}^n} e^{-iO^*x\xi}\varphi(x)dx = \begin{bmatrix} y = O^*x \\ x = Oy \end{bmatrix} \\ &= \int_{\mathbb{R}^n} e^{-iy\xi}\varphi(Oy)dy = (\varphi \circ O)^\wedge(\xi).\end{aligned}$$

This implies

$$\begin{aligned}&\langle \widehat{u} \circ O, \varphi\rangle = \langle \widehat{u}, \varphi \circ O^{-1}\rangle = \langle u, (\varphi \circ O^*)^\wedge\rangle \\ &= \langle u, \widehat{\varphi} \circ O^*\rangle = \langle u, \widehat{\varphi} \circ O^{-1}\rangle = \langle u \circ O, \widehat{\varphi}\rangle = \langle u, \widehat{\varphi}\rangle = \langle \widehat{u}, \varphi\rangle.\end{aligned}$$

□

Theorem 12.11. *If u is a tempered distribution that is homogeneous of degree α, then $\widehat{u}$ is homogeneous of degree $-n-\alpha$.*

Proof. By Definition 4.4, $u \in \mathscr{S}'$ is homogeneous of degree α if $\langle u, \varphi_t\rangle = t^\alpha \langle u, \varphi\rangle$. Therefore,

$$\langle \widehat{u}, \varphi_t\rangle = \langle u, \widehat{\varphi}_t\rangle = \langle u_\xi, \widehat{\varphi}(t\xi)\rangle = t^{-n}\langle u, (\widehat{\varphi})_{1/t}\rangle$$
$$= t^{-(n+\alpha)}\langle u, \widehat{\varphi}\rangle = t^{-(n+\alpha)}\langle \widehat{u}, \varphi\rangle.$$

□

Example 12.12. A fundamental solution of the Laplace operator when $n \geq 3$.

By Fourier transformation of

$$\Delta u = \delta,$$

we get

$$-|\xi|^2 \widehat{u}(\xi) = 1.$$

One solution is

$$\widehat{u}(\xi) = -\frac{1}{|\xi|^2}.$$

Observe that $\frac{1}{|\xi|^2} \in L^1_{\text{loc}}(\mathbb{R}^n)$ if $n \geq 3$, and that $\frac{1}{|\xi|^2}$ is radial and homogeneous of degree-2. Hence u is radial and homogeneous of degree $2-n$. This implies

$$u(x) = \frac{c_n}{|x|^{n-2}}.$$

Argument. If $n = 3$, then $\frac{1}{|\xi|^2} \in L^1 + L^2$, and thus u is a function. If $n > 4$, then $\frac{1}{|x|^{n-2}} \in L^1 + L^2$, and we can argue as above using the inversion theorem. When $n = 4$, we have $u_\epsilon = \dfrac{1}{|x|^{2+\epsilon}} \in L^1 + L^2$, and thus its Fourier transform is a constant times $\dfrac{1}{|\xi|^{2-\epsilon}}$ and the statement follows by letting $\epsilon \to 0$.

An alternative way is to use Exercise 4 below.

Exercise 12.2. What is c_n

Exercise 12.3. What happens if $n = 2$?

Exercise 12.4. Determine all radial distributions in $\mathbb{R}^n$ that are homogeneous of degree α.

Hint. Consider first $n = 1$. Compute the derivate of $\langle u, \varphi_t\rangle = t^\alpha\langle u, \varphi\rangle$ with respect to t.

Warning. Be careful when $-\alpha = n, n+2, n+4, \ldots$.

Exercise 12.5. What is the Fourier transform of $\mathrm{fp}|x|^\alpha$ in $\mathbb{R}$?

Exercise 12.6. What is a reasonably definition of $\mathrm{fp}|x|^\alpha$ in $\mathbb{R}^n$? What is its Fourier transform?

Exercise 12.7. Determine a fundamental solution to the heat equation.

Hint. Determine $\mathcal{F}E(x,t)$, where $\mathcal{F}$ is the Fourier transform with respect to $x \in \mathbb{R}^n$.

Theorem 12.13. *If $u \in \mathscr{S}'$ and $v \in \mathscr{E}'$, then $u * v \in \mathscr{S}'$ and*

$$(u * v)^\wedge = \widehat{v}\,\widehat{u}.$$

Proof. If $\varphi \in C_0^\infty$, then

$$u * v(\varphi) = (u * v) * \overset{\vee}{\varphi}(0) = u * (v * \overset{\vee}{\varphi})(0) = u((v * \overset{\vee}{\varphi})^\vee) = u(\overset{\vee}{v} * \varphi).$$

To see that $u * v \in \mathscr{S}'$, we need to show that $\overset{\vee}{v} * \varphi_j \to 0$ in $\mathscr{S}$ when $\varphi_j \to 0$ in $\mathscr{S}$. Let K be a compact neigborhood of av supp v and k the order of v. Then

$$|\partial^\beta(\overset{\vee}{v} * \varphi_j)(x)| = |\langle v_y, \partial^\beta \varphi_j(y - x)\rangle| \le C \sum_{|\gamma| \le k} \sup_{y \in K} |\partial^{\beta+\gamma}\varphi_j(x - y)|.$$

Thus

$$(1+|x|^2)^\ell |\partial^\beta(\overset{\vee}{v} * \varphi_j)(x)| \le C \sum_{|\gamma| \le k} (1+|x|^2)^\ell \sup_{y \in K} |\partial^{\beta+\gamma}\varphi_j(x-y)| \to 0, \quad j \to \infty.$$

To compute the Fourier transform, we observe that

$$\begin{aligned}(u * v)^\wedge(\varphi) &= u * v(\widehat{\varphi}) = u(\overset{\vee}{v} * \widehat{\varphi}) \\ &= (2\pi)^{-n} u(\widehat{\widehat{v}} * \widehat{\varphi}) = u((\widehat{v}\varphi)^\wedge) = \widehat{u}(\widehat{v}\varphi) = \widehat{v}\,\widehat{u}(\varphi).\end{aligned}$$

□

Exercise 12.8. Compute $(\frac{1}{1+x^2})^{*n}$ and $(e^{-x^2})^{*n}$.

Exercise 12.9. H 7.1.6

Exercise 12.10. H 7.1.7

Exercise 12.11. H 7.1.9

Exercise 12.12. H 7.1.11

Exercise 12.13. H 7.1.18

Exercise 12.14. H 7.1.28

Chapter 13

The Paley-Wiener theorem

If $u \in \mathscr{E}'(\mathbb{R}^n)$, we know that $\hat{u} \in C^\infty$ and that

$$\hat{u}(\xi) = u(e^{-ix\xi}).$$

We shall show that $\hat{u}$ can be extended to an analytic function in $\mathbb{C}^n$, ie. $\hat{u}(\zeta_1, \ldots, \zeta_n)$ is an analytic function in each variable $\zeta_1, \ldots, \zeta_n$. We start with a version of the theorem for test functions.

Proposition 13.1.

(a) *If $\phi \in C_0^\infty$ and supp $\phi \subset \{x; |x| \leq R\}$, then*

$$\hat{\phi}(\zeta) = \int_{\mathbb{R}^n} e^{-ix\zeta}\phi(x)dx$$

is an entire function with

$$|\hat{\phi}(\zeta)| \leq C_N(1 + |\zeta|)^{-N} e^{R|\operatorname{Im}\zeta|} \tag{13.1}$$

for all N.

(b) *Conversely, if $\hat{\phi}$ is entire and satifies (1), then $\phi \in C_0^\infty$ and supp $\phi \subset \{x; |x| \leq R\}$.*

Proof. (a) By differentiation under the integral sign, we see that $\hat{\phi}$ is analytic. Furthermore, if $\zeta = \xi + i\eta$,

$$|\hat{\phi}(\zeta)| \leq \int_{|x|\leq R} e^{x\eta}|\phi(x)|dx \leq Ce^{R|\eta|}\ . \tag{13.2}$$

If we apply (2) to $D^\alpha\phi$, (1) follows.

(b) Since $\hat{\phi}$ is rapidly decreasing we can differentiate under the integral sign in the Fourier inversion formula. Hence $\phi \in C^\infty$.

Again, by the rapid decrease of $\hat{\phi}$ we can use the Cauchy theorem to change the contour of integration and integrate along $\{\zeta; \mathrm{Im}\zeta_i = \eta_i\}$. We get

$$|\phi(x)| = \left|(2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix(\xi+i\eta)} \hat{\phi}(\xi + i\eta) d\xi\right| \leq Ce^{-x\eta} e^{R|\eta|}.$$

If we let $\eta = tx$, we obtain

$$|\phi(x)| \leq Ce^{-t|x|(|x|-R)}.$$

Thus if $|x| > R$, and we let $t \to \infty$, we obtain $\phi(x) = 0$. Hence supp $\phi \subset \{x; |x| \leq R\}$. □

For distributions we have

Theorem 13.2 (The Paley-Wiener thorem)**.**

(a) *If u is a distribution of order N with support in $\{x; |x| \leq R\}$, then $\hat{u}$ is an entire function and*

$$|\hat{u}(\zeta)| \leq C(1 + |\zeta|)^N e^{R\,|\mathrm{Im}\,\zeta|}. \tag{13.3}$$

(b) *Conversely, if $\hat{u}$ is an entire function that satisfies* (3) *for some N, then u is a distribution that is supported in $\{x; |x| \leq R\}$.*

Proof. (a) That $\hat{u}$ is entire follows since

$$\frac{\partial}{\partial \zeta_i} \hat{u}(\zeta) = \frac{\partial}{\partial \zeta_i} u(e^{-ix\zeta}) = u\Big(\frac{\partial}{\partial \zeta_i}(e^{-ix\zeta})\Big).$$

The last equality holds as

$$\frac{e^{-ix(\zeta+\omega_i)} - e^{-ix\zeta}}{\omega_i} \to \frac{\partial}{\partial \zeta_i}(e^{-ix\zeta}) \text{ i } C^\infty, \quad \omega_i \to 0.$$

To prove (3), we fix $\chi_\delta \in C_0^\infty$ with $\chi_\delta = 1$ in a neighborhood of $\{x; |x| \leq R\}$ and supp $\chi_\delta \subset \{x; |x| < R + \delta\}$. We can choose χ_δ such that $\|D^\alpha \chi_\delta\|_\infty \leq C\delta^{-|\alpha|}$. We obtain

$$\begin{aligned} |\hat{u}(\zeta)| &= |u(e^{-ix\zeta})| = |u(\chi_\delta(x) e^{-ix\zeta})| \\ &\leq C_N \sup \sum_{|\alpha| \leq N} |D_x^\alpha(\chi_\delta(x) e^{-ix\zeta})| \\ &\leq Ce^{(R+\delta)|\mathrm{Im}\zeta|} \sum_{|\beta| \leq N} \delta^{-|\beta|}(1 + |\zeta|)^{N-\beta}. \end{aligned}$$

If we let $\delta = \frac{1}{1+|\zeta|}$, (3) follows.

(b) The polynomial growth of $\hat{u}$ implies that $\hat{u}$, and hence also u, is in $\mathscr{S}'$. Let $\varphi_\delta \in \mathscr{S}$ be an approximative identity and let $u_\delta = u * \varphi_\delta$. Then $u_\delta \in C^\infty$, $u_\delta \to u$ as $\delta \to 0$, and

$$|\hat{u}_\delta(\zeta)| = |\hat{u}(\zeta)\hat{\varphi}(\delta\zeta)| \leq C_{M,\delta}(1+|\zeta|)^{-M} \exp((R+c\delta)|\mathrm{Im}\zeta|).$$

Here we have used (3) and (1) in Proposition 1. If we apply Proposition 1 to u_δ, we get supp $u_\delta \subset \{x; |x| \leq (R+c\delta)\}$. If we let $\delta \to 0$, we obtain supp $u \subset \{x; |x| \leq R\}$. □

Exercise 13.1. Assume that $u, v \in \mathscr{E}'(\mathbb{R}^n)$ and that $u * v = 0$. Show that then $u = 0$ or $v = 0$. What happens if only one of u and v have compact support?

Exercise 13.2. H 7.1.40.

Chapter 14

Existence of fundamental solutions

Let $P(D)$ be a differential operator with constant coefficients in $\mathbb{R}^n$. We shall show that $P(D)$ has a fundamental solution E.

Let us first make a formal computation. By Fourier transformation of $P(D)E = \delta$, we get $P(\xi)\widehat{E}(\xi) = 1$ and $\widehat{E}(\xi) = P(\xi)^{-1}$. Now

$$\langle E, \varphi \rangle = \langle E, \check{\check{\varphi}} \rangle = (2\pi)^{-n} \langle E, \widehat{\check{\varphi}} \rangle = (2\pi)^{-n} \langle \widehat{E}, \check{\widehat{\varphi}} \rangle \ .$$

Hence it is natural to define E by

$$\langle E, \varphi \rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} P(\xi)^{-1} \widehat{\varphi}(-\xi) d\xi.$$

Then (formally)

$$\begin{aligned} \langle P(D)E, \varphi \rangle &= \langle E, P(-D)\varphi \rangle = (2\pi)^{-n} \int_{\mathbb{R}^n} P(\xi)^{-1} (P(-D)\varphi)\widehat{\ }(-\xi) d\xi \\ &= (2\pi)^{-n} \int_{\mathbb{R}^n} P(\xi)^{-1} P(\xi) \widehat{\varphi}(-\xi) d\xi = \varphi(0) = \langle \delta, \varphi \rangle. \end{aligned}$$

However, this does not always work since $P(\xi)$ may vanish. Therefore, we will change the contour of integration and define $\langle E, \varphi \rangle$ by an integral along a set in $\mathbb{C}^n$ that contains no zero of P.

Theorem 14.1. *Every linear differential operator with constant coefficients has a fundamental solution $E \in \mathscr{D}'$.*

Proof. Let $m = \text{grad } P$. After a linear change of variables P is of the form

$$\begin{aligned} P(\xi) &= P_{\xi'}(\xi_n) = \xi_n^m + P_{m-1}(\xi')\xi_n^{m-1} + \ldots + P_0(\xi') \\ &= (\xi_n - \alpha_1(\xi')) \ldots (\xi_n - \alpha_m(\xi')). \end{aligned}$$

Here $\xi = (\xi_1, \dots, \xi_n) = (\xi', \xi_n)$ and $\alpha_i(\xi')$ are the zeros of $P_{\xi'}(\xi_n)$. We can choose $\phi(\xi') \in \mathbb{R}$ such that $|\phi(\xi')| \leq m+1$ and $|\phi(\xi') - \alpha_i(\xi')| \geq |\phi(\xi') - \text{Im}\ \alpha_i(\xi')| \geq 1$ for $i = 1, 2, \dots, m$. Define $\langle E, \varphi \rangle$, when $\varphi \in \mathscr{D}$, by

$$\langle E, \varphi \rangle = (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} d\xi' \int_{\text{Im}\ \zeta_n = \phi(\xi')} P(\zeta)^{-1} \widehat{\varphi}(-\zeta) d\zeta_n.$$

By the Paley-Wiener theorem, $\widehat{\varphi}(\zeta)$ is an entire function and

$$|\widehat{\varphi}(\zeta)| \leq \frac{C}{(1+|\zeta|)^N} \sum_{|\alpha| \leq N} \|D^\alpha \varphi\|_\infty.$$

Furthermore $|P(\zeta)^{-1}| \leq 1$, and hence, if N is large enough, we get

$$|\langle E, \varphi \rangle| \leq C \sum_{|\alpha| \leq N} \|D^\alpha \varphi\|_\infty.$$

Thus $E \in \mathscr{D}'$. Finally, we see that

$$\begin{aligned}
\langle P(D)E, \varphi \rangle &= \langle E, P(-D)\varphi \rangle \\
&= (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} d\xi' \int_{\text{Im}\ \zeta_n = \phi(\xi')} P(\zeta)^{-1} (P(-D)\varphi)^\wedge(-\zeta) d\zeta_n \\
&= (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} d\xi' \int_{\text{Im}\ \zeta_n = \phi(\xi')} \widehat{\varphi}(-\zeta) d\zeta_n \\
&= \text{the Cauchy theorem} = (2\pi)^{-n} \int_{\mathbb{R}^{n-1}} d\xi' \int_{\mathbb{R}} \widehat{\varphi}(-\xi) d\xi_n \\
&= (2\pi)^{-n} \int_{\mathbb{R}^n} \widehat{\varphi}(\xi) d\xi = \varphi(0) = \langle \delta, \varphi \rangle.
\end{aligned}$$

□

Exercise 14.1. Determine a fundamental solution to the Schrödinger equation

$$(D_t - \sum_1^n D_{x_i}^2) E = \delta.$$

$(D = -i\partial)$
Hint. See Exercise 10.14 and the hint to Excercise 12.7

Chapter 15

Fundamental solutions of elliptic differential operators

Let $P(D)$ be a differential operator with constant coefficients. We write the polynomial P as

$$P = P_m + P_{m-1} + \ldots + P_0,$$

where P_k is a homogeneous polynomial of degree k. The operator $P(D)$ is called *elliptic* if $P_m(\xi) \neq 0$ for $\xi \neq 0$, $\xi \in \mathbb{R}^n$.

Example 15.1. Δ and $\bar{\partial}$ are elliptic. The heat and wave operators are not elliptic. □

Theorem 15.1. *Let $P(D)$ be an elliptic differential operator Then there is a distribution $E \in \mathscr{S}'(\mathbb{R}^n)$ such that* $\operatorname{sing\,supp} E = \{0\}$ *and $P(D)E = \delta - \omega$, for some $\omega \in \mathscr{S}(\mathbb{R}^n)$.*

Corollary 15.2. *If P is elliptic, then P is hypoelliptic.*

Proof of the corollary. We shall show that u is C^∞ if $P(D)u$ is. If u has compact support, we have $u = \delta * u = (P(D)E + \omega) * u = E * P(D)u + \omega * u \in C^\infty$. The general case follows by considering $\psi_n u$ where $\psi_n \in C_0^\infty$ with $\psi_n = 1$ on $\{|x| \leq n\}$ (compare Theorem 9.6) □

Proof of the theorem. Since P is elliptic, $|P_m(\xi)| \geq \delta > 0$ when $|\xi| = 1$. By homogenity this implies

$$|P_m(\xi)| \geq \delta |\xi|^m.$$

Hence, if $|\xi| > R$ where R is large enough,

$$|P(\xi)| \geq c|\xi|^m.$$

Take $\chi \in C_0^\infty(\mathbb{R}^n)$ with $\chi(\xi) = 1$ if $|\xi| \leq R$. Then $(1-\chi)P^{-1}$ is bounded and hence a tempered distribution. Thus we can define $E \in \mathscr{S}'(\mathbb{R}^n)$ by

$$\widehat{E} = \frac{1-\chi}{P}.$$

Then,

$$(P(D)E)^\wedge = P\widehat{E} = P\frac{1-\chi}{P} = 1-\chi = \widehat{\delta} - \chi.$$

If we define ω by $\widehat{\omega} = \chi$, then $\omega \in \widehat{\mathscr{D}} \subset \mathscr{S}$ and $P(D)E = \delta - \omega$. It remains to show that $E \in C^\infty(\mathbb{R}^n \setminus \{0\})$. Observe that

$$(x^\beta D^\alpha E)^\wedge(\xi) = cD^\beta(\xi^\alpha \frac{1-\chi(\xi)}{P(\xi)}) = O(|\xi|^{-|\beta|-m+|\alpha|}), \quad |\xi| \to \infty.$$

If we choose $|\beta|$ large enough, we get $(x^\beta D^\alpha E)^\wedge \in L^1$. Thus $x^\beta D^\alpha E \in C$ and hence $D^\alpha E \in C(\mathbb{R}^n \setminus \{0\})$, and the proof is complete. □

Chapter 16

Fourier series

Let u be a distribution that is periodic with period 2π in each variable, i.e.

$$\langle u, \tau_{2\pi k}\varphi\rangle = \langle u, \varphi\rangle,$$

if $k \in \mathbb{Z}^n$. Intuitively u is determined by its "values" on

$$T = \{x; 0 \leq x_i < 2\pi\}.$$

Lemma 16.1. *If u is periodic, then $u \in \mathscr{S}'$.*

Proof. Let $\psi \in C_0^\infty$ with $0 \leq \psi \leq 1$ and $\psi = 1$ on T. Put

$$\widetilde{\psi}(x) = \sum_{k\in\mathbb{Z}^n} \psi(x - 2\pi k).$$

Then $\widetilde{\psi}$ is a periodic C^∞-function with $\widetilde{\psi} \geq 1$. Thus $\phi = \psi/\widetilde{\psi} \in C_0^\infty$ and

$$\sum_k \phi(x - 2\pi k) = 1.$$

If $\varphi \in \mathscr{D}$, then

$$\begin{aligned}\langle u, \varphi\rangle &= \langle u_x, \sum_k \phi(x - 2\pi k)\varphi(x)\rangle = \text{a finite sum} =\\ &= \sum_k \langle u_x, \phi(x - 2\pi k)\varphi(x)\rangle = \text{periodicity} =\\ &= \sum_k \langle u_x, \phi(x)\varphi(x + 2\pi k)\rangle = \langle u_x, \phi(x) \sum_k \varphi(x + 2\pi k)\rangle.\end{aligned}$$

But if $\varphi_j \to 0$ in $\mathscr{S}$, then $\phi(x)\sum_k \varphi_j(x+2\pi k) \to 0$ i $\mathscr{D}$. (Prove that!) Hence the right hand side defines an extension of u to $\mathscr{S}'$. $\square$

To compute $\widehat{u}$, we first show the following result.

Theorem 16.2 (The Poisson summation formula)**.** *If $\varphi \in \mathscr{S}$, then*

$$\sum_{k\in\mathbb{Z}^n} \widehat{\varphi}(2\pi k) = \sum_{k\in\mathbb{Z}^n} \varphi(k).$$

Proof. Let $u = \sum_{k\in\mathbb{Z}^n} \delta_{2\pi k}$. Then $\delta_{2\pi l} * u = u$, since $\delta_{2\pi l} * \delta_{2\pi k} = \delta_{2\pi(k+l)}$. (Prove that!)
Hence

$$(e^{-2\pi i l\xi} - 1)\widehat{u} = 0.$$

But $e^{-2\pi i l\xi} - 1 \neq 0$ if $\xi \notin \mathbb{Z}^n$, and consequently $\widehat{u}$ is supported on $\mathbb{Z}^n$. By choosing different l, we see that close to the origin we have $\xi_i \widehat{u} = 0$, $i = 1, 2, \dots, n$. Thus $\widehat{u} = c\delta_0$ there. Furthermore $e^{-ikx}u = u$, and hence $\widehat{u}$ is invariant under translation by integers. From this, we obtain

$$\widehat{u} = c \sum_{k\in\mathbb{Z}^N} \delta_k.$$

This means that

$$\sum_{k\in\mathbb{Z}^n} \widehat{\varphi}(2\pi k) = c \sum_{k\in\mathbb{Z}^n} \varphi(k).$$

If we replace φ with a translation of φ, we get

$$\sum_{k\in\mathbb{Z}^n} \widehat{\varphi}(2\pi k) e^{2\pi i k x} = c \sum_{k\in\mathbb{Z}^n} \varphi(k + x).$$

Integration over $\{x; 0 \le x_i < 1\}$ gives

$$\widehat{\varphi}(0) = c \int_{\mathbb{R}^n} \varphi(x)dx = c\widehat{\varphi}(0)\ .$$

Thus $c = 1$ and the proof is complete. □

Let us return to the computation of $\widehat{u}$ when u is periodic. Using the Poisson summation formula on $\varphi(y) = \psi(y)e^{-ixy}$, we get, as $\widehat{\varphi}(\xi) = \widehat{\psi}(x+\xi)$,

$$\sum_k \widehat{\psi}(x + 2\pi k) = \sum_k \widehat{\varphi}(2\pi k) = \sum_k \varphi(k) = \sum_k e^{-ixk}\psi(k).$$

From the proof of Lemma 1, we have

$$\begin{aligned} \langle \widehat{u}, \psi \rangle &= \langle u, \widehat{\psi} \rangle = \langle u, \phi(x) \sum_k \widehat{\psi}(x + k) \rangle \\ &= \langle u, \phi(x) \sum_k e^{-ixk}\psi(k) \rangle \\ &= \sum_k \psi(k) \langle u, \phi(x) e^{-ixk} \rangle. \end{aligned}$$

Hence $\widehat{u} = \sum_k c_k \delta_k$,where

$$c_k = \langle u, \phi(x) e^{-ixk} \rangle.$$

In particular, if u is an integrable function on T, we have

$$\begin{aligned} c_k = \langle u, \phi(x) e^{-ixk} \rangle &= \int_{\mathbb{R}^n} u(x)\phi(x) e^{-ixk} dx \\ &= \sum_j \int_T u(x - 2\pi j)\phi(x - 2\pi j) e^{-i(x-2\pi j)k} dx \\ &= \int_T u(x) e^{-ixk} \sum_j \phi(x - 2\pi j) dx = \int_T u(x) e^{-ixk} dx. \end{aligned}$$

Hence c_k are "our old" Fourier coefficients. The inversion theorem implies that

$$u(x) = \frac{1}{(2\pi)^n} \sum_k c_k e^{ikx} \quad \text{in} \quad \mathscr{S}'.$$

If $u \in C^l$, then $c_k = O(|k|^{-l}), |k| \to \infty$, and the sum is uniformly convergent if $l > n$. Thus we have proved

Theorem 16.3. *If $u \in C^l(\mathbb{R}^n)$, $l > n$, and u is periodic with period 2π in each variable, then*

$$u(x) = \frac{1}{(2\pi)^n} \sum_k c_k e^{ixk},$$

where the series is uniformly convergent.

We finish this chapter by proving

Theorem 16.4 (The Plancherel theorem)**.** *If $u \in L^2(T)$ with Fourier coefficients c_k, then*

$$u(x) = \frac{1}{(2\pi)^n} \sum c_k e^{ixk} \text{ in } L^2,$$

and

$$\int_T |u|^2 dx = \frac{1}{(2\pi)^n} \sum |c_k|^2.$$

Conversely, if $\sum |c_k|^2 < \infty$, then $u(x) = \frac{1}{(2\pi)^n} \sum_k c_k e^{ixk}$ is a function in $L^2(T)$ with Fourier coefficients c_k.

Proof. If $u \in C^{n+1}$, the series is uniformly convergent, and we get

$$\int_T |u|^2 dx = \frac{1}{(2\pi)^{2n}} \sum_{k,l} c_k \bar{c}_l \int_T e^{ix(k-l)} dx = \frac{1}{(2\pi)^n} \sum |c_k|^2.$$

As C^{n+1} is dense in L^2, we can extend this to $u \in L^2$: Take $u_n \in C^{n+1}$, $u_n \to u$ in L^2. Then, also $u_n \to u$ in $\mathscr{S}'$ and $\widehat{u}_n \to \widehat{u}$ in $\mathscr{S}'$. But also, by the isometry, $\widehat{u}_n$ is a Cauchy sequence in l^2. This implies $\widehat{u}_n \to \widehat{u}$ in l^2. Hence

$$\int_T |u|^2 dx = \lim_{n\to\infty} \int_T |u_n|^2 dx = \lim_{n\to\infty} \frac{1}{(2\pi)^n} \sum_k |c_k(u_n)|^2 = \frac{1}{(2\pi)^n} \sum |c_k|^2.$$

Conversely, if $\sum |c_k|^2 < \infty$, let

$$u(x) = \frac{1}{(2\pi)^n} \sum_k c_k e^{ixk} \text{ and } u_N(x) = \frac{1}{(2\pi)^n} \sum_{|k|\leq N} c_k e^{ixk}.$$

Then, $u_N \to u$ in L^2 and $\mathscr{S}'$, and we get

$$\widehat{u} = \lim_{N\to\infty} \widehat{u}_N = \sum c_k \delta_k.$$

□

Remark 16.5. If u is a function with period t, then $u_t(x) = u(\frac{2\pi x}{t})$ has period 2π. Using this, we can generalise Fourier series to functions with arbitrary period. □

Exercise 16.1. H 7.2.1

Exercise 16.2. H 7.2.5

Exercise 16.3. H 7.2.8

Exercise 16.4. Compute a) $\sum_{-\infty}^{\infty} \frac{1}{1+n^2}$ b) $\sum_{-\infty}^{+\infty} \frac{1}{(n+a)^2}$ and c) $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3}$.

Chapter 17

Some applications

17.1 The central limit theorem

Let $X, X_1, X_2, \ldots$ be independent identically distributed stochastic variables with $E[X] = m$ and $Var[X] = \sigma^2$. Then

$$\lim_{n\to\infty} P\Big(\frac{X_1 + X_2 + \ldots + X_n - nm}{\sigma\sqrt{n}} \le x\Big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}y^2} dy. \tag{17.1}$$

Some background: To a stochastic variable X we associate a probability measure μ on $\mathbb{R}$ (we write $X \sim \mu$) by

$$P(X \le x) = \int_{-\infty}^{x} d\mu(y).$$

If μ_1 are μ_2 probability measures, we define a new probability measure $\mu_1 * \mu_2$ by

$$\langle \mu_1 * \mu_2, \varphi\rangle = \iint_{\mathbb{R}^2} \varphi(x+y) d\mu_1(x) d\mu_2(y).$$

Then $(\mu_1 * \mu_2)^\wedge = \widehat{\mu}_1 \widehat{\mu}_2$. (Show that!) If $X \sim \mu_1$ and $Y \sim \mu_2$ are independent, then $X + Y \sim \mu_1 * \mu_2$.

Proof. We may assume that $m = 0$ and $\sigma = 1$. Let

$$S_n = \frac{X_1 + \ldots + X_n}{\sqrt{n}}$$

and

$$\mu^{n*} = \underbrace{\mu * \ldots * \mu}_{n \text{ times}}.$$

Then $S_n \sim \mu_n$, where

$$\langle \mu_n, \varphi \rangle = \int_{\mathbb{R}} \varphi \left(\frac{x}{\sqrt{n}} \right) d\mu^{n*}(x),$$

and

$$\widehat{\mu_n}(\xi) = \left(\widehat{\mu} \left(\frac{\xi}{\sqrt{n}} \right) \right)^n .$$

Since $\mathrm{Var}[X] < \infty$,

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} d\mu(x)$$

is a C^2-function with

$$\widehat{\mu}\,'(0) = -im = 0 \quad \text{and} \quad \widehat{\mu}\,''(0) = -\sigma^2 = -1.$$

Thus

$$\widehat{\mu}(\xi) = 1 - \frac{1}{2}\xi^2 + o(\xi^2), \quad \xi \to 0,$$

and

$$\widehat{\mu_n}(\xi) = \left(\widehat{\mu} \left(\frac{\xi}{\sqrt{n}} \right) \right)^n = \left(1 - \frac{1}{2n}\xi^2 + o\left(\frac{\xi^2}{n} \right) \right)^n \to e^{-\frac{1}{2}\xi^2}, \quad n \to \infty,$$

for each fixed ξ. But, since $|\widehat{\mu}(\xi)| \leq 1$, we get, by dominated convergence, that

$$\widehat{\mu_n}(\xi) \to e^{-\frac{1}{2}\xi^2} \quad \text{in} \quad \mathscr{S}'.$$

Hence Fourier inversion implies that

$$\mu_n \to \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

in $\mathscr{S}'$, and hence also in $\mathscr{D}'$. But μ_n are positive measures, and by Theorem 7.4

$$\mu_n \to \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

weakly, and hence we obtain (1). □

17.2 The mean value property for harmonic functions

If $u \in C^\infty$ is harmonic in a neighborhood of $\{|x| \leq 1\}$, then

$$u(0) = \frac{1}{\omega_n} \int_{S^{n-1}} u(y) d\sigma y).$$

Remark 17.1. By Weyl's lemma, the assumption that $u \in C^\infty$ is unnecessary. □

Proof. Define a distribution Λ by

$$\langle \Lambda, \varphi \rangle = \int_{S^{n-1}} \varphi(y) d\sigma(y) - \omega_n \varphi(0).$$

Then $\Lambda \in \mathscr{E}'$, and hence $\widehat{\Lambda}$ is an entire function. Furthermore, Λ, and therefore also $\widehat{\Lambda}$, is radial. Hence $\widehat{\Lambda}(\zeta) = G(|\zeta|)$, where $G(t) = \widehat{\Lambda}(t, 0, \ldots, 0)$ is holomorphic. Also, G is even, so $G(z) = F(z^2)$ for some entire function F. Since $F(0) = \widehat{\Lambda}(0) = \Lambda(1) = 0$,

$$\frac{\widehat{\Lambda}(\xi)}{|\xi|^2} = \frac{F(|\xi|^2) - F(0)}{|\xi|^2}$$

is the restriction of an entire function. By the Paley-Wiener theorem, there is a distribution $\mu \in \mathscr{E}'$ with $\widehat{\mu}(\xi) = -F(|\xi|^2)/|\xi|^2$, and so

$$(\Delta\mu)\widehat{\ }(\xi) = -|\xi|^2 \widehat{\mu}(\xi) = \widehat{\Lambda}(\xi).$$

Hence $\Delta\mu = \Lambda$, which gives

$$\langle \Lambda, u \rangle = \langle \Delta\mu, u \rangle = \langle \mu, \Delta u \rangle = \langle \mu, 0 \rangle = 0.$$

□

17.3 The Heisenberg uncertainty principle

If $f \in L^2(\mathbb{R})$, then

$$\|xf(x)\|_2 \|\xi \widehat{f}(\xi)\|_2 \geq \sqrt{\frac{\pi}{2}} \|f\|_2^2, \tag{17.2}$$

with equality only if $f(x) = \exp(-kx^2)$, $k > 0$.

Quantum mechanical background: The state of a particle is described by a function $\psi \in L^2(\mathbb{R})$ with $\|\psi\|_2 = 1$. We interprete

$$\int_E |\psi|^2$$

as the probability that the particle is in the set E. An observable quantity A is a symmetric operator on a suitable subspace of L^2. The mean value of A in the state ψ is

$$E[A] = \int A\psi \cdot \bar{\psi} = \langle A\psi, \psi \rangle.$$

That A is symmetric means that $A = A^*$ and hence we have

$$\langle A\psi, \psi\rangle = \langle \psi, A^*\psi\rangle = \langle \psi, A\psi\rangle = \overline{\langle A\psi, \psi\rangle}.$$

Thus the mean value is real.

Example 17.2.
a) Position. $A\psi(x) = x\psi(x)$
b) Momentum. $B\psi = 2\pi i\psi'$. □

We have

$$E[B] = \int B\psi \cdot \bar{\psi} = 2\pi i \int \psi'\bar{\psi} = \text{Plancherel} = \int \xi\widehat{\psi}(\xi)\overline{\psi(\xi)} = \int \xi|\widehat{\psi}(\xi)|^2 d\xi.$$

Hence we can interprete $|\widehat{\psi}(\xi)|^2$ as the density of the momentum.

The general form of the Heisenberg uncertainty principle is

$$E[(A - E[A])^2]E[(B - E[B])^2] \geq \frac{1}{4}\big|E[AB - BA]\big|^2 \tag{17.3}$$

for arbitrary A and B.

Exercise 17.1. Show that if A and B are position and momentum, then $AB - BA = -2\pi i$.

Exercise 17.2. Prove that (2) implies (3), when A and B are position and momentum.

Proof. If $f \in \mathscr{S}$, then

$$\|xf(x)\|_2\|\xi\widehat{f}(\xi)\|_2 = \|xf(x)\|_2\|\widehat{f'}(\xi)\|_2 = \text{Parseval} =$$
$$= \sqrt{2\pi}\|xf(x)\|_2\|f'(x)\|_2 \geq \text{Schwartz} \geq \sqrt{2\pi}\int |x\overline{f(x)}f'(x)|dx$$
$$\geq (|x\bar{z}w| \geq x \text{ Re } \bar{z}w) \geq \sqrt{2\pi}\int x\frac{1}{2}\left(\overline{f(x)}f'(x) + f(x)\overline{f'(x)}\right)dx$$
$$= \sqrt{\frac{\pi}{2}}\int x(|f(x)|^2)'dx = \text{Integration by parts} = \sqrt{\frac{\pi}{2}}\int |f|^2dx = \sqrt{\frac{\pi}{2}}\|f\|_2^2.$$

The proof that the theorem holds for functions in L^2, and the statement of equality is left to the reader. □

17.4 A primer on Sobolev inequalities

A benefit of the theory of distributions is that we can find solutions to problems that has no classical solutions. But we often want our solutions to be nice functions. Therefore it is natural to ask the question

When is a distributional solution a function?

The theory of Sobolev spaces gives us a method to answer that question.

We start with the simplest result in Sobolev theory,

The Sobolev L^1-inequality *Let f be an integrable function on $\mathbb{R}^n$. Assume that the distributional derivatives $\partial^\alpha f$ also are integrable for all $|\alpha| \leq n$. Then f is a bounded continuous function and*

$$\|f\|_\infty \leq \sum_{|\alpha|\leq n} \|\partial^\alpha f\|_1 \,. \tag{17.4}$$

If furthermore $\partial^\alpha f$ are integrable for all $|\alpha| \leq n+k$, then f is a C^k-function.

Proof. We start with the case $n = 1$ where we shall show that

$$\|f\|_\infty \leq \|f\|_1 + \|f'\|_1 \,. \tag{17.5}$$

If $\varphi \in C_0^\infty$, then

$$|\varphi(x)| = \left| \int_{-\infty}^{x} \varphi'(t)dt \right| \leq \int_{-\infty}^{x} |\varphi'(t)|dt \leq \int_{-\infty}^{\infty} |\varphi'(t)|dt \,.$$

This implies that

$$\|\varphi\|_\infty \leq \|\varphi'\|_1 \,. \tag{17.6}$$

This inequality is sharper than (5), but we have obtained it asssuming two strong extra conditions, C^∞ and compact support. (The function 1 shows that (6) can not be true in general.)

If $f \in C^\infty$ is not compactly supported, we choose a sequence of cut off functions $\chi_n \in C^\infty$, with $\chi_n = 1$ as $|x| \leq n$ and $\|\chi_n'\|_\infty \leq 1$. If we apply (6) to $\varphi = \chi_n f$,we get

$$\|\chi_n f\|_\infty \leq \|(\chi_n f)'\|_1 \leq \|\chi_n' f\|_1 + \|\chi_n f'\|_1 \leq \|f\|_1 + \|f'\|_1 \,.$$

Since n is arbitrary, (5) is proved for C^∞-functions.

If f is not C^∞, we let ϕ_δ be an approximative identity. Then $f_\delta = \phi_\delta * f \in C^\infty$ and we can apply (5) to f_δ. We get, as $\|\phi_\delta * f\|_1 \leq \|f\|_1$ och $\|(\phi_\delta * f)'\|_1 = \|\phi_\delta * f'\|_1 \leq \|f'\|_1$, that

$$\|\phi_\delta * f\|_\infty \leq \|f\|_1 + \|f'\|_1 \,.$$

But $\phi_\delta * f \to f$ a.e. and we have proved (5) in the general case.

To finish the proof, we apply (5) to $f - f_\delta$, to obtain

$$\|f - f_\delta\|_\infty \le \|f - \phi_\delta * f\|_1 + \|f' - \phi_\delta * f'\|_1 \to 0, \delta \to 0 \ .$$

Thus $f_\delta \to f$ uniformly, and f is a continuous function.

The last claim follows by applying this argument to the functions $\partial^i f$, $i = 1, 2, \ldots, k$.

The argument when $n \ge 2$ is similar. The case $n = 2$ shows how but without too cumbersome notation. If $\varphi \in C_0^\infty$, we now get

$$|\varphi(x,y)| = \left| \int_\infty^x \int_\infty^y \partial^{(1,1)} \varphi(s,t) ds dt \right| \le \int_\infty^\infty \int_\infty^\infty |\partial^{(1,1)} \varphi(s,t)| ds dt$$

and hence

$$\|\varphi\|_\infty \le \|\partial^{(1,1)} \varphi\|_1 \ .$$

When we apply this to $\chi_n f$, $f \in C^\infty$, we get, as $\partial^{(1,1)}(\chi_n f) = \partial^{(1,1)} \chi_n \, f + \partial^{(1,0)} \chi_n \partial^{0,1} f + \partial^{(0,1)} \chi_n \partial^{1,0} f + \chi_n \partial^{(1,1)} f$, that

$$\|f\|_\infty \le \|f\|_1 + \|\partial^{1,0} f\|_1 + \|\partial^{0,1} f\|_1 + \|\partial^{(1,1)} f\|_1, \ f \in C^\infty \ .$$

The rest of the argument works exactly the same as in the case $n = 1$.

□

Remark 17.3. The proof shows that it is enough to consider $\alpha = (\alpha_1, \ldots, \alpha_n)$, where each α_i is either 0 or 1, in the sum (4). □

The Sobolev L^2-inequality *Let $f \in \mathscr{D}'(\Omega)$, where Ω is an open set in $\mathbb{R}^n$, and let r and $k \ge 0$ be integers. If $\partial^\alpha f \in L^2_{\text{loc}}$ for all α, $0 \le |\alpha| \le r$ where $r > k + \frac{n}{2}$, then $f \in C^k(\Omega)$.*

Proof when $n = 1$ and $k = 0$.

The assumption means that $f \in L^2_{\text{loc}}$ and $f' \in L^2_{\text{loc}}$. Let ω be an open set, $\omega \subset\subset \Omega$, and take $\chi \in C_0^\infty(\Omega)$ with $\|\chi\|_\infty \le 1$, $\|\chi'\|_\infty \le 1$ and $\chi = 1$ in a neighborhood of ω. Define $F(x) = F_\omega(x) = \chi(x) f(x)$. ($F = 0$ outside the support of χ.) Since $F \in L^2(\mathbb{R})$ and $F' = \chi' f + \chi f' \in L^2(\mathbb{R})$, the Parseval identity implies that

$$\int_{\mathbb{R}} |\widehat{F}|^2 d\xi < \infty \ \text{ and } \ \int_{\mathbb{R}} \xi^2 |\widehat{F}|^2 d\xi < \infty \ .$$

Hence

$$\int_{\mathbb{R}} (1+|\xi|)^2 |\widehat{F}|^2 d\xi < \infty .$$

The Cauchy-Schwartz inequality implies

$$\left(\int_{\mathbb{R}} |\widehat{F}| d\xi\right)^2 = \left(\int_{\mathbb{R}} (1+|\xi|)|\widehat{F}| \frac{d\xi}{1+|\xi|}\right)^2$$
$$\leq \int_{\mathbb{R}} (1+|\xi|)^2 |\widehat{F}|^2 d\xi \int_{\mathbb{R}} \frac{d\xi}{(1+|\xi|)^2} < \infty .$$

Thus $\widehat{F}$ is integrable and hence F is continuous. As ω is an arbitrary open subset of Ω, it follows that $f \in C(\Omega)$. □

The general case.

If $n = 1$ and $k = 1$, we also know that $F'' \in L^2$. Thus $\xi^2 \widehat{F}(\xi) \in L^2$ and we have $\int_{\mathbb{R}} (1+|\xi|)^4 |\widehat{F}|^2 d\xi < \infty$. By the Cauchy inequality, this gives $\int_{\mathbb{R}} (1+|\xi|)|\widehat{F}| d\xi < \infty$. In particular, $\xi \widehat{F}(\xi)$ is integrable and F' is continuous. The case for arbitrary k follows in the same way.

If $n > 1$, the condition on $\partial^\alpha f$ implies that $\xi_i^l \widehat{F}(\xi) \in L^2$, $l \leq r$. Using the inequality $(1+|\xi|)^{2l} \leq C_l(1+\xi_1^{2l}+\ldots+\xi_n^{2l})$, and the Cauchy inequality, we obtain $(1+|\xi|)^k \widehat{F} \in L^1$ and hence $F \in C^k$.

□

Sobolev spaces

Let us abstract the ideas in the proof of the L^2-inequality. We saw that if f and its derivatives up to order r are in L^2, then $(1+|\xi|)^r \widehat{f} \in L^2$ or equivalently $(1+|\xi|^2)^{r/2} \widehat{f} \in L^2$. In this condition, r may be an arbitary real number, and we can make the following definition.

Definition 17.4. A distribution $f \in \mathscr{D}'(\mathbb{R}^n)$, is in the *Sobolev space* $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, if

$$\|f\|_{H^s} = \|(1+|\xi|^2)^{s/2} \widehat{f}(\xi)\|_{L^2} < +\infty$$

Proposition 17.5. *If $f \in H^r(\mathbb{R}^n)$, where $r > s + \frac{n}{2}$, then* $(1+|\xi|)^s \widehat{f} \in L^1(\mathbb{R}^n)$.

Remark 17.6. If $s = k$ is a non-negative integer, this implies that $f \in C^k$.

Proof. We have $(1+|\xi|^2)^{r/2}\widehat{f}(\xi) \in L^2$. Hence, by the Cauchy inequality, we get

$$\begin{aligned}\int_{\mathbb{R}^n} &(1+|\xi|^2)^{s/2}\widehat{f}(\xi)d\xi \\ &= \int_{\mathbb{R}^n}(1+|\xi|^2)^{r/2}\widehat{f}(\xi)\frac{d\xi}{(1+|\xi|^2)^{(r-s)/2}} \leq C\|f\|_{H^r} \ ,\end{aligned}$$

as $(1+|\xi|^2)^{-(s-r)}$ is integrable when $r-s>\frac{n}{2}$. □

17.5 Minkowski's theorem

Let B be a convex set in $\mathbb{R}^n$ that is symmetric at the origin. If $|B|\geq 2^n$, then B contains more than one lattice point.

Proof. We assume that 0 is the only lattice point in B, and show that this implies that $|B|<2^n$. Let $f=\chi_B*\chi_B$. Since B is symmetric, $\widehat{\chi}_B$ is real. Hence $\widehat{f}=(\widehat{\chi}_B)^2=|\widehat{\chi}_B|^2$.

We observe that if $f(2k)\neq 0$, ie.

$$f(2k)=\int \chi_B(2k-x)\chi_B(x)dx \neq 0 \ ,$$

then there is $x\in B$ with $2k-x\in B$. But then we have $k=\frac{1}{2}(2k-x)+\frac{1}{2}x\in B$, as B is convex. Hence if $f(2k)\neq 0$ we have $k=0$. Furthermore

$$f(0)=\int \chi_B(-x)\chi_B(x)dx=\int |\chi_B|^2dx=|B|.$$

The Poisson summation formula, applied to the lattices $(2\mathbb{Z})^n$ and $(\pi\mathbb{Z})^n$, gives

$$\sum_{j\in\mathbb{Z}^n} f(2j)=2^{-n}\sum_{j\in\mathbb{Z}^n}\widehat{f}(\pi j)\ .$$

Hence

$$\begin{aligned}|B| = f(0) &= \sum_j f(2j) = 2^{-n}\sum_j \widehat{f}\,(\pi j) \\ &= 2^{-n}\sum_j |\widehat{\chi}_B\,(\pi j)|^2 = 2^{-n}\left(|B|^2+\sum_{j\neq 0}|\widehat{\chi}_B\,(\pi j)|^2\right)\ .\end{aligned}$$

If we can show that

$$\sum_{j\neq 0}|\widehat{\chi}_B\,(\pi j)|^2>0,$$

we obtain $|B| > 2^{-n}|B|^2$, or $|B| < 2^n$, and we are done.

But if $\widehat{\chi}_B(\pi j) = 0$ when $j \neq 0$, then

$$\chi(x) = \sum_j \chi_B(x + 2j)$$

is constant. This follows from the Poisson summation formula since

$$\chi(x) = \sum_j \tau_{-x}\chi_B(2j) = 2^{-n}\sum_j e^{i\pi xj}\widehat{\chi}_B(\pi j) = 2^{-n}\widehat{\chi}_B(0).$$

But this is a contradiction as

$$\chi(0) = 1 \neq 0 = \chi(1, 0, \ldots, 0).$$

□

Exercise 17.3. The proof is "wrong". Why? Correct it!

Index