MA 342H
Assignment 3
Due 29 March 2018

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1. Compute the Euler-Lagrage equations for

   (a) the first order Lagrangian in one independent variable

   $$L(x, u, u_x) = x\sqrt{1 + u_x^2},$$

   (b) the second order Lagrangian in one independent variable

   $$L(x, u, u_x, u_{xx}) = \frac{u_{xx}}{1 + u_x^2},$$

   and

   (c) the first order Lagrangian in three independent variables

   $$L(t, x, y, u, u_t, u_x, u_y) = -\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + \frac{1}{2}u^2.$$

   Solution:

   (a) The relevant derivatievs are

   $$\frac{\partial L}{\partial u} = 0, , \quad \frac{\partial L}{\partial u_x} = \frac{xu_x}{\sqrt{1 + u_x^2}}, \quad D_x\frac{\partial L}{\partial u_x} = \frac{xu_{xx} + u_x^3 + u_x}{(1 + u_x^2)^{3/2}}$$

   and the Euler-Lagrange equation is

   $$\frac{\partial L}{\partial u} - D_x\frac{\partial L}{\partial u_x} = -\frac{xu_{xx} + u_x^3 + u_x}{(1 + u_x^2)^{3/2}} = 0.$$

(b) The relevant derivatives are

$$\frac{\partial L}{\partial u} = 0,$$

$$\frac{\partial L}{\partial u_x} = -2\frac{u_x u_{xx}}{(1+u_x^2)^2},$$

$$\frac{\partial L}{\partial u_{xx}} = \frac{1}{1+u_x^2},$$

$$D_x\frac{\partial L}{\partial u_{xx}} = -2\frac{u_x u_{xx}}{(1+u_x^2)^2},$$

$$\frac{\partial L}{\partial u} - D_x\frac{\partial L}{\partial u_x} + D_x^2\frac{\partial L}{\partial u_{xx}} = \frac{\partial L}{\partial u} - D_x\left(\frac{\partial L}{\partial u_x} - D_x\frac{\partial L}{\partial u_{xx}}\right) = 0 - 0 = 0.$$

So the Euler-Lagrange equation is just $0 = 0$. In other words L is a null Lagrangian. That makes sense, because

$$L = D_x \arctan(u_x).$$

(c) The relevant derivatives are

$$\frac{\partial L}{\partial u} = u, \quad \frac{\partial L}{\partial u_t} = -u_t, \quad \frac{\partial L}{\partial u_x} = u_x, \quad \frac{\partial L}{\partial u_y} = u_y,$$

$$D_t\frac{\partial L}{\partial u_t} = -u_{tt}, \quad D_x\frac{\partial L}{\partial u_x} = -u_{xx}, \quad D_y\frac{\partial L}{\partial u_y} = -u_{yy},$$

so the Euler-Lagrange equation is

$$\frac{\partial L}{\partial u} - D_t\frac{\partial L}{\partial u_t} - D_x\frac{\partial L}{\partial u_x} - D_y\frac{\partial L}{\partial u_y} = u_{tt} - u_{xx} - u_{yy} + u$$

2. Find the vector fields associated to the one parameter groups of transformations on $\mathbf{R}^3 \times \mathbf{R}$, with coordinates t, x, y, u,

   (a)

   $$\tilde{t} = t\cosh s + x\sinh s, \quad \tilde{x} = t\sinh s + x\cosh s, \quad \tilde{y} = y, \quad \tilde{u} = u$$

   and

   (b)

   $$\tilde{t} = e^{2s}t, \quad \tilde{x} = e^{2s}x, \quad \tilde{y} = e^{2s}y, \quad \tilde{u} = e^{-s}u.$$

Solution: In both cases the associated vector field is

$$\left.\frac{\partial\tilde{t}}{\partial s}\right|_{s=0}\frac{\partial}{\partial t}+\left.\frac{\partial\tilde{x}}{\partial s}\right|_{s=0}\frac{\partial}{\partial x}+\left.\frac{\partial\tilde{y}}{\partial s}\right|_{s=0}\frac{\partial}{\partial y}+\left.\frac{\partial\tilde{u}}{\partial s}\right|_{s=0}\frac{\partial}{\partial u}.$$

With the given transformations we get

(a)

$$x\frac{\partial}{\partial t}+t\frac{\partial}{\partial x}$$

and

(b)

$$2t\frac{\partial}{\partial t}+2x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}-u\frac{\partial}{\partial u}$$

3. Find the first order prolongations of the vector fields from the previous problem.

Solution: The general formula for the first order prolongation of

$$\sum_{i=1}^{m}\xi_i\frac{\partial}{\partial x_i}+\sum_{j=1}^{n}\eta_j\frac{\partial}{\partial u_j}$$

is

$$\sum_{i=1}^{m}\xi_i\frac{\partial}{\partial x_i}+\sum_{j=1}^{n}\eta_j\frac{\partial}{\partial u_j}+\sum_{i=1}^{m}\sum_{j=1}^{n}\left(D_i\eta_j-\sum_{l=1}^{m}(D_i\xi_l)u_{j,l}\right)\frac{\partial}{u_{j,i}}$$

In both examples here we have $m=3$, $n=1$ and we can, for clarity, replace the indices $1,2,3$ for ξ by t,x,y and drop the index 1 for η. The general formula then tells us that the first order prolongation of

$$\xi_t\frac{\partial}{\partial t}+\xi_x\frac{\partial}{\partial x}+\xi_y\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial u}$$

is

$$\begin{aligned}\xi_t\frac{\partial}{\partial t}&+\xi_x\frac{\partial}{\partial x}+\xi_y\frac{\partial}{\partial y}+\eta\frac{\partial}{\partial u}\\&+(D_t\eta-(D_t\xi_t)u_t-(D_t\xi_x)u_x-(D_t\xi_y)u_y)\frac{\partial}{\partial u_t}\\&+(D_x\eta-(D_x\xi_t)u_t-(D_x\xi_x)u_x-(D_x\xi_y)u_y)\frac{\partial}{\partial u_x}\\&+(D_y\eta-(D_y\xi_t)u_t-(D_y\xi_x)u_x-(D_y\xi_y)u_y)\frac{\partial}{\partial u_y}.\end{aligned}$$

For
$$x\frac{\partial}{\partial t}+t\frac{\partial}{\partial x}$$
we have
$$\xi_t=x,\quad \xi_x=t,\quad \xi_y=0,\quad \eta=0$$
and we get
$$x\frac{\partial}{\partial t}+t\frac{\partial}{\partial x}-u_x\frac{\partial}{\partial u_t}-u_t\frac{\partial}{\partial u_x}.$$
For
$$2t\frac{\partial}{\partial t}+2x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}-u\frac{\partial}{\partial u}$$
we have
$$\xi_t=2t,\quad \xi_x=2x,\quad \xi_y=2y,\quad \eta=-u$$
and we get
$$2t\frac{\partial}{\partial t}+2x\frac{\partial}{\partial x}+2y\frac{\partial}{\partial y}-u\frac{\partial}{\partial u}-3u_t\frac{\partial}{\partial u_t}-3u_x\frac{\partial}{\partial u_x}-3u_y\frac{\partial}{\partial u_y}.$$

4. Show that the vector fields above are symmetries of the first order Lagrangian
$$L(t,x,y,u,u_t,u_x,u_y)=-\frac{1}{2}u_t^2+\frac{1}{2}u_x^2+\frac{1}{2}u_y^2$$
and that the first, but not the second, of them is also a symmetry of
$$L(t,x,y,u,u_t,u_x,u_y)=-\frac{1}{2}u_t^2+\frac{1}{2}u_x^2+\frac{1}{2}u_y^2+\frac{1}{2}u^2.$$
Solution: For
$$V=x\frac{\partial}{\partial t}+t\frac{\partial}{\partial x}$$
the prolonged vector field
$$\operatorname{pr}^{(1)}V=x\frac{\partial}{\partial t}+t\frac{\partial}{\partial x}-u_x\frac{\partial}{\partial u_t}-u_t\frac{\partial}{\partial u_x},$$
applied to the Lagrangian
$$L=-\frac{1}{2}u_t^2+\frac{1}{2}u_x^2+\frac{1}{2}u_y^2$$
gives zero. The total divergence
$$\operatorname{Div}V=D_t\xi_t+D_x\xi_x+D_y\xi_y=D_tx+D_xt$$

is also zero, so

$$(\mathrm{pr}^{(1)} V)L + (\mathrm{Div}\, V)L = 0.$$

For

$$V = 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}$$

the prolonged vector field

$$\mathrm{Div}\, V = 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u} - 3u_t\frac{\partial}{\partial u_t} - 3u_x\frac{\partial}{\partial u_x} - 3u_y\frac{\partial}{\partial u_y}.$$

applied to L gives

$$(\mathrm{pr}^{(1)} V)L = 3u_t^2 - 3u_x^2 - 3u_y^2 = -6L.$$

The total divergence is

$$\mathrm{Div}\, V = D_t\xi_t + D_x\xi_x + D_y\xi_y = D_t(2t) + D_x(2x) + D_y(2y) = 6,$$

so again

$$(\mathrm{pr}^{(1)} V)L + (\mathrm{Div}\, V)L = 0.$$

The calculation for

$$L = -\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + \frac{1}{2}u^2$$

is identical, except that $\partial L/\partial u$ is no longer zero. For

$$V = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}$$

nothing changes, but for

$$V = 2t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial y} - u\frac{\partial}{\partial u}$$

we acquire an additional $-u^2$ in $(\mathrm{pr}^{(1)} V)L$, and an additional $3u^2$ in $(\mathrm{Div}\, V)L$, so

$$(\mathrm{pr}^{(1)} V)L + (\mathrm{Div}\, V)L = 2u^2 \neq 0.$$

5. Find the conservation laws associated to

   (a) The transformation

   $$\tilde{t} = t\cosh s + x\sinh s, \quad \tilde{x} = t\sinh s + x\cosh s, \quad \tilde{y} = y, \quad \tilde{u} = u$$

   for the Lagrangian

   $$L(t, x, y, u, u_t, u_x, u_y) = -\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2 + \frac{1}{2}u^2,$$

   and

(b) the transformation

$$\tilde{t} = e^{2s}t, \quad \tilde{x} = e^{2s}x, \quad \tilde{y} = e^{2s}y, \quad \tilde{u} = e^{-s}u.$$

for the Lagrangian

$$L(t, x, y, u, u_t, u_x, u_y) = -\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2 + \frac{1}{2}u_y^2.$$

Solution: In general for a first order Lagrangian L and a vector field

$$\sum_{i=1}^{m} \xi_i \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} \eta_j \frac{\partial}{\partial u_j}$$

associated to a symmetry of L we have the conserved current

$$P_i = \xi_i L + \sum_{j=1}^{n} \left(\eta_j - \sum_{l=1}^{m} u_{j,l}\xi_l \right) \frac{\partial L}{\partial u_{j,i}}.$$

In our context this becomes

$$P_t = \xi_t L + (\eta - u_t\xi_t - u_x\xi_x - u_y\xi_y) \frac{\partial L}{\partial u_t},$$

$$P_x = \xi_x L + (\eta - u_t\xi_t - u_x\xi_x - u_y\xi_y) \frac{\partial L}{\partial u_x},$$

$$P_y = \xi_y L + (\eta - u_t\xi_t - u_x\xi_x - u_y\xi_y) \frac{\partial L}{\partial u_y}.$$

For the given transformations and Lagrangians we get

(a)

$$\begin{aligned}
P_t &= x\left(-\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_y^2 + \tfrac{1}{2}u^2\right) \\
&\quad + (-xu_t - tu_x)(-u_t) \\
&= \tfrac{1}{2}x\left(u_t^2 + u_x^2 + u_y^2 + u^2\right) + tu_tu_x, \\
P_x &= t\left(-\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_y^2 + \tfrac{1}{2}u^2\right) \\
&\quad + (-xu_t - tu_x)(u_x) \\
&= -\tfrac{1}{2}t\left(u_t^2 + u_x^2 - u_y^2 - u^2\right) - xu_tu_x, \\
P_y &= 0\left(-\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_y^2 + \tfrac{1}{2}u^2\right) \\
&\quad + (-xu_t - tu_x)(u_y) \\
&= -xu_tu_y - tu_xu_y,
\end{aligned}$$

and

(b)

$$
\begin{aligned}
P_t &= 2t\left(-\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_y^2\right) \\
&\quad + \left(-u - 2tu_t - 2xu_x - 2yu_y\right)\left(-u_t\right) \\
&= t\left(u_t^2 + u_x^2 + u_y^2\right) + 2xu_tu_x + 2yu_tu_y + uu_t, \\
P_x &= 2x\left(-\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_y^2\right) \\
&\quad + \left(-u - 2tu_t - 2xu_x - 2yu_y\right)\left(u_x\right) \\
&= -2tu_tu_x - x\left(u_t^2 + u_x^2 - u_y^2\right) - 2yu_xu_y - uu_x, \\
P_y &= 2y\left(-\tfrac{1}{2}u_t^2 + \tfrac{1}{2}u_x^2 + \tfrac{1}{2}u_y^2\right) \\
&\quad + \left(-u - 2tu_t - 2xu_x - 2yu_y\right)\left(u_y\right) \\
&= -2tu_tu_y - 2xu_xu_y - y\left(u_t^2 - u_x^2 + u_y^2\right) - uu_y.
\end{aligned}
$$