MA 342H
Assignment 2
Due 14 March 2018

`Id: 342H-2017-2018-2.m4,v 1.1 2018/03/07 14:36:26 jgs Exp jgs`

1. Show that
$$k(x) = -\frac{\exp(-\|x\|)}{4\pi\|x\|}$$
is a fundamental solution for
$$p(\partial) = \partial_1^2 + \partial_2^2 + \partial_3^2 - 1$$
in $\mathbf{R}^3$.
Hint: Use Green's second identity on the region
$$\int_\Omega (u \operatorname{div} \operatorname{grad} v - v \operatorname{div} \operatorname{grad} u) = \int_{\partial\Omega} (u \operatorname{grad} v - v \operatorname{grad} u) \cdot n$$
where $\partial\Omega$ is the boundary of Ω and n is the exterior unit normal there. You'll want
$$\Omega = \{x \in \mathbf{R}^3 \colon \epsilon < \|x\| < \rho\}.$$
$u = k$ and $v = \varphi \in \mathcal{D}(\mathbf{R}^3)$. *Solution:*
We choose ρ large enough that φ is supported in the open ball of radisc of radius ρ. We need to show that
$$p(\partial)k = \delta$$
or, equivalently, that
$$\int_{\mathbf{R}^3} k(x)(\operatorname{div} \operatorname{grad} \varphi(x) - \varphi(x)) = k(p(\partial)\varphi) = \delta(\varphi) = \varphi(0).$$

A straightforward calculation shows that

$$\operatorname{grad} k(x) = \frac{\exp(-\|x\|)x}{4\pi\|x\|^2} + \frac{\exp(-\|x\|)x}{4\pi\|x\|^3}$$

and

$$\operatorname{div}\operatorname{grad} k(x) = -\frac{\exp(-\|x\|)}{4\pi\|x\|} = k(x)$$

for $x \neq 0$. It follows that

$$\begin{aligned}\int_\Omega (k \operatorname{div}\operatorname{grad}\varphi - \varphi \operatorname{div}\operatorname{grad} k) &= \int_\Omega (k(\operatorname{div}\operatorname{grad}\varphi - \varphi) - \varphi(\operatorname{div}\operatorname{grad} k - k)) \\ &= \int_\Omega k(\operatorname{div}\operatorname{grad}\varphi - \varphi).\end{aligned}$$

As ϵ tends to 0 this tends to the integral over the ball of radius ρ, but that's the same as the integral over all of $\mathbf{R}^3$, since φ and its derivatives vanish outside that ball. So

$$\lim_{\epsilon\to 0^+} \int_\Omega (k \operatorname{div}\operatorname{grad}\varphi - \varphi \operatorname{div}\operatorname{grad} k) = \int_{\mathbf{R}^3} k(\operatorname{div}\operatorname{grad}\varphi - \varphi).$$

Turning now to the integral

$$\int_{\partial\Omega} (k \operatorname{grad}\varphi - \varphi \operatorname{grad} k)\cdot n,$$

φ and $\operatorname{grad}\varphi$ are zero on the sphere $\|x\| = \rho$ so the integral over that part of $\partial\Omega$ is zero. On the remaining part of $\partial\Omega$, the sphere $\|x\| = \epsilon$, we have

$$n = -\frac{x}{\|x\|},$$

$$k(x) = -\frac{\exp(-\epsilon)}{4\pi\epsilon},$$

and

$$\operatorname{grad} k(x)\cdot n = -\frac{\exp(-\epsilon)}{4\pi\epsilon} - \frac{\exp(-\epsilon)}{4\pi\epsilon^2}.$$

So

$$\begin{aligned}\int_{\partial\Omega} (k \operatorname{grad}\varphi - \varphi \operatorname{grad} k)\cdot n = &\frac{\exp(-\epsilon)}{4\pi\epsilon^2}\int_{\|x\|=\epsilon} \varphi(x) \\ &+ \frac{\exp(-\epsilon)}{4\pi\epsilon}\int_{\|x\|=\epsilon} \varphi(x) \\ &- \frac{\exp(-\epsilon)}{4\pi\epsilon}\int_{\|x\|=\epsilon} \operatorname{grad}\varphi(x)\cdot n\end{aligned}$$

In the last two integrals the integrands are bounded uniformly by $\max|\varphi|$ and $\max\|\operatorname{grad}\varphi\|$ and the sphere over which we're integrating is of area $4\pi\epsilon^2$ so

$$\left|\frac{\exp(-\epsilon)}{4\pi\epsilon}\int_{\|x\|=\epsilon}\varphi(x)\right| \leq \epsilon\max|\varphi|$$

and

$$\left|\frac{\exp(-\epsilon)}{4\pi\epsilon}\int_{\|x\|=\epsilon}\right| \leq \epsilon\max\|\operatorname{grad}\varphi\|,$$

both of which tend to zero as ϵ tends to 0. The first integral, which is just $\exp(-\epsilon)$ times the average value of φ over the sphere $\|x\| = \epsilon$ tends to $\varphi(0)$, so

$$\lim_{\epsilon\to 0^+}\int_{\partial\Omega}(k\operatorname{grad}\varphi - \varphi\operatorname{grad}k)\cdot n = \varphi(0)$$

Taking the limit in Green's identity therefore gives the desired result

$$\int_{\mathbf{R}^3} k(x)(\operatorname{div}\operatorname{grad}\varphi(x) - \varphi(x)) = \varphi(0).$$

2. (a) Find a fundamental solution for the differential operator

$$p(\partial) = \partial^2 + 2\partial + 2$$

on $\mathbf{R}$.

(b) Use the fundamental solution you just found to solve the initial value problem for the inhomogeneous equation

$$u''(x) + 2u'(x) + 2u(x) = f(x), \quad u(0) = \alpha, \quad u'(0) = \beta.$$

If you didn't manage to find a fundamental solution then just take as given that there is one and call it k.

Solution:

(a) We want

$$p(\partial)k = \delta$$

and hence

$$(p(\partial)k)\star\varphi = \delta\star\varphi = \varphi.$$

For such k we have

$$k\star(p(\partial)\varphi) = \varphi$$

and
$$\hat{k}(\xi)p(i\xi)\hat{\varphi}(\xi) = \hat{\varphi}(\xi).$$
Since this should hold for all $\varphi \in (\mathbf{R})$ we must have
$$\hat{k}(\xi) = p(i\xi)^{-1}$$
and hence
$$k(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{k}(\xi) \exp(i\xi x)\, d\xi = \int_{-\infty}^{\infty} g(\xi)\, d\xi,$$
where
$$g(\xi) = (2\pi)^{-1} p(i\xi)^{-1} \exp(i\xi x).$$
Note that g is analytic except for simple poles at $1+i$ and $-1+i$. For $R > \sqrt{2}$ let C_1 be the contour from $-\infty$ to $-R$ along the real axis, C_2 the contour from $-R$ to R along the real axis, C_3 the contour from R to $+\infty$ along the real axis, C_4 the contour from R to $-R$ along the circle $|\xi| = R$ in the upper half-plane and C_4 the contour from $-R$ to R along the circle $|\xi| = R$ in the lower half-plane. Then
$$\begin{aligned}\left|\int_{C_1} g(\xi)\, d\xi\right| &\le \int_{C_1} |g(\xi)|\, d\xi \\ &= \int_{-\infty}^{-R} (2\pi)^{-1}(\xi^4+4)^{-1/2}\, d\xi \\ &\le \int_{-\infty}^{-R} (2\pi)^{-1}\xi^{-2}\, d\xi \\ &= (2\pi R)^{-1}\end{aligned}$$
and
$$\begin{aligned}\left|\int_{C_3} g(\xi)\, d\xi\right| &\le \int_{C_3} |g(\xi)|\, d\xi \\ &= \int_{R}^{\infty} (2\pi)^{-1}(\xi^4+4)^{-1/2}\, d\xi \\ &\le \int_{R}^{\infty} (2\pi)^{-1}\xi^{-2}\, d\xi \\ &= (2\pi R)^{-1}.\end{aligned}$$
On the circle $|\xi| = R$ we have
$$\left|p(i\xi)^{-1}\right| = |\xi - 1 - i|^{-1}\, |\xi - 1 + i|^{-1} \le (R - \sqrt{2})^{-2}.$$

If $x \geq 0$ then
$$|\exp(i\xi x)| \leq 1$$
for ξ in the upper half-plane and hence
$$|g(\xi)| \leq (2\pi)^{-1}(R - \sqrt{2})^{-2}$$
and
$$\left|\int_{C_4} g(\xi)\, d\xi\right| \leq \frac{R}{2(R-\sqrt{2})^2}.$$
If $x \leq 0$ then
$$|\exp(i\xi x)| \leq 1$$
for ξ in the lower half-plane and hence
$$|g(\xi)| \leq (2\pi)^{-1}(R - \sqrt{2})^{-2}$$
and
$$\left|\int_{C_5} g(\xi)\, d\xi\right| \leq \frac{R}{2(R-\sqrt{2})^2}.$$
By the Residue Theorem
$$\begin{aligned}\int_{C_2} g(\xi)\, d\xi + \int_{C_4} g(\xi)\, d\xi &= 2\pi i \operatorname*{Res}_{\xi=-1+i} g(\xi) + 2\pi i \operatorname*{Res}_{\xi=1+i} g(\xi) \\ &= \frac{\exp(i(-1+i)x)}{2i} - \frac{\exp(i(1+i)x)}{2i} \\ &= e^{-x}\sin x\end{aligned}$$
and
$$-\int_{C_2} g(\xi)\, d\xi + \int_{C_5} g(\xi)\, d\xi = 0.$$
If $x \geq 0$ we write
$$\begin{aligned}k(x) &= \int_{C_1} g(\xi)\, d\xi + \int_{C_2} g(\xi)\, d\xi + \int_{C_3} g(\xi)\, d\xi \\ &= \int_{C_1} g(\xi)\, d\xi + \int_{C_3} g(\xi)\, d\xi - \int_{C_4} g(\xi)\, d\xi \\ &\quad + \int_{C_2} g(\xi)\, d\xi + \int_{C_4} g(\xi)\, d\xi\end{aligned}$$
and take the limit as $R \to \infty$ to get
$$k(x) = e^{-x}\sin x.$$

If $x \leq 0$ we write

$$\begin{aligned}
k(x) &= \int_{C_1} g(\xi)\, d\xi + \int_{C_2} g(\xi)\, d\xi + \int_{C_3} g(\xi)\, d\xi \\
&= \int_{C_1} g(\xi)\, d\xi + \int_{C_3} g(\xi)\, d\xi + \int_{C_5} g(\xi)\, d\xi \\
&\quad + \int_{C_2} g(\xi)\, d\xi - \int_{C_5} g(\xi)\, d\xi
\end{aligned}$$

and take the limit as $R \to \infty$ to get

$$k(x) = 0.$$

So the fundamental solution is

$$k(x) = \begin{cases} e^{-x} \sin x & \text{if } x \geq 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

(b) Let H be the Heaviside function

$$H(x) = \begin{cases} 1 & \text{if } x > 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then

$$\begin{gathered}
\partial H = \delta, \\
\partial(uH) = (\partial u)H + u(\partial H) = u'H + u\delta = u'H + u(0)\delta \\
\partial^2(uH) = u''H + u'(0)\delta + u(0)\partial\delta, \\
(\delta^2 + 2\delta + 2)(uH) = (u'' + 2u' + 2u)H + (u'(0) + 2u(0))\delta + u(0)\partial\delta.
\end{gathered}$$

If u satisfies our initial value problem then

$$(\delta^2 + 2\delta + 2)(uH) = fH + (\beta + 2\alpha)\delta + \alpha\partial\delta.$$

$$\begin{aligned}
uH &= \delta \star (uH) = (p(\partial)k) \star (uH) = k \star (p(\partial)(uH) \\
&= k \star (fH + (\beta + 2\alpha)\delta + \alpha\partial\delta) = k \star fH + (\beta + 2\alpha)k + \alpha\partial k
\end{aligned}$$

If $x > 0$ then

$$\begin{aligned}
u(x) &= u(x)H(x) \\
&= \int_{-\infty}^{\infty} k(x-y)f(y)H(y)\, dy + (\beta + 2\alpha)k(x) + \alpha k'(x) \\
&= \int_0^{\infty} k(x-y)f(y)\, dy + (\beta + 2\alpha)k(x) + \alpha k'(x)
\end{aligned}$$

With the particular fundamental solution found above

$$u(x) = \int_0^x e^{-(x-y)} \sin(x-y) f(y)\, dy + (\beta + \alpha)e^{-x} \sin x + \alpha e^{-x} \cos x.$$