MA 342H
Assignment 1
Due 14 February 2018

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1. For the linear first order scalar partial differential equation

$$\begin{aligned}(-x-y-z)\frac{\partial u}{\partial w}(w,x,y,z)+(w-y+z)\frac{\partial u}{\partial x}(w,x,y,z)\\ +(w+x-z)\frac{\partial u}{\partial y}(w,x,y,z)+(w-x+y)\frac{\partial u}{\partial z}(w,x,y,z)=0,\end{aligned}$$

with initial conditions

$$u(1,x,y,z)=f(x,y,z)$$

(a) Find the non-characteristic points on the initial hypersurface.

(b) Write down and solve the characteristic equations.

(c) Solve the differential equation. Note that you only need your solution to make sense near the non-characteristic part of the initial hypersurface.

Solution:

(a) We can do this either using the implicit form

$$g(w,x,y,z)=w-1$$

of the initial hypersurface or the parameterisation

$$w=1,\quad x=q,\quad y=r,\quad z=s.$$

The former is a bit easier. We need

$$g(w,x,y,z)=0$$

and

$$\begin{aligned}(-x-y-z)\frac{\partial g}{\partial w}(w,x,y,z)+(w-y+z)\frac{\partial g}{\partial x}(w,x,y,z)\\ +(w+x-z)\frac{\partial g}{\partial y}(w,x,y,z)+(w-x+y)\frac{\partial g}{\partial z}(w,x,y,z)\neq 0.\end{aligned}$$

For the given g this means

$$w=1, \quad x+y+z\neq 0.$$

If we are using the parameterisation then we need

$$\det\begin{pmatrix}0 & 0 & 0 & -q-r-s\\ 1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\end{pmatrix}\neq 0,$$

which leads to the same result.

(b) The characteristic equations are

$$w'=-x-y-z, \quad x'=w-y+z, \quad y'=w+x-z, \quad z'=w-x+y,$$

$$u'=0$$

or, in matrix form

$$\frac{\partial}{\partial t}\begin{pmatrix}w\\ x\\ y\\ z\end{pmatrix}=\begin{pmatrix}0 & -1 & -1 & -1\\ 1 & 0 & -1 & 1\\ 1 & 1 & 0 & -1\\ 1 & -1 & 1 & 0\end{pmatrix}\begin{pmatrix}w\\ x\\ y\\ z\end{pmatrix}$$

$$u'=0$$

The initial conditions are

$$\begin{pmatrix}w\\ x\\ y\\ z\end{pmatrix}=\begin{pmatrix}1\\ q\\ r\\ s\end{pmatrix}, \quad u=f(q,r,s)$$

when $t = 0$. The solution is

$$\begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{3}t) & -\frac{\sin(\sqrt{3}t)}{\sqrt{3}} & -\frac{\sin(\sqrt{3}t)}{\sqrt{3}} & -\frac{\sin(\sqrt{3}t)}{\sqrt{3}} \\ \frac{\sin(\sqrt{3}t)}{\sqrt{3}} & \cos(\sqrt{3}t) & -\frac{\sin(\sqrt{3}t)}{\sqrt{3}} & \frac{\sin(\sqrt{3}t)}{\sqrt{3}} \\ \frac{\sin(\sqrt{3}t)}{\sqrt{3}} & \frac{\sin(\sqrt{3}t)}{\sqrt{3}} & \cos(\sqrt{3}t) & -\frac{\sin(\sqrt{3}t)}{\sqrt{3}} \\ \frac{\sin(\sqrt{3}t)}{\sqrt{3}} & -\frac{\sin(\sqrt{3}t)}{\sqrt{3}} & \frac{\sin(\sqrt{3}t)}{\sqrt{3}} & \cos(\sqrt{3}t) \end{pmatrix} \begin{pmatrix} 1 \\ q \\ r \\ s \end{pmatrix}$$

$$u = f(q, r, s).$$

(c) We need to use the equations from the previous part to eliminate q, r, s and t. The matrix above is orthogonal, since it came from exponentiating an antisymmetric matrix. It follows that

$$\begin{pmatrix} 1 \\ q \\ r \\ s \end{pmatrix} = \begin{pmatrix} \cos(\sqrt{3}t) & \frac{\sin(\sqrt{3}t)}{\sqrt{3}} & \frac{\sin(\sqrt{3}t)}{\sqrt{3}} & \frac{\sin(\sqrt{3}t)}{\sqrt{3}} \\ -\frac{\sin(\sqrt{3}t)}{\sqrt{3}} & \cos(\sqrt{3}t) & \frac{\sin(\sqrt{3}t)}{\sqrt{3}} & -\frac{\sin(\sqrt{3}t)}{\sqrt{3}} \\ -\frac{\sin(\sqrt{3}t)}{\sqrt{3}} & -\frac{\sin(\sqrt{3}t)}{\sqrt{3}} & \cos(\sqrt{3}t) & \frac{\sin(\sqrt{3}t)}{\sqrt{3}} \\ -\frac{\sin(\sqrt{3}t)}{\sqrt{3}} & \frac{\sin(\sqrt{3}t)}{\sqrt{3}} & -\frac{\sin(\sqrt{3}t)}{\sqrt{3}} & \cos(\sqrt{3}t) \end{pmatrix} \begin{pmatrix} w \\ x \\ y \\ z \end{pmatrix}$$

The first row gives the equation

$$1 = \cos(\sqrt{3}t)w + \sin(\sqrt{3}t)\frac{x + y + z}{\sqrt{3}}.$$

Making the rationalising substitution

$$v = \tan\left(\frac{\sqrt{3}}{2}t\right), \quad \cos(\sqrt{3}t) = \frac{1 - v^2}{1 + v^2}, \quad \sin(\sqrt{3}t) = \frac{2v}{1 + v^2}$$

gives

$$(1 + v^2) = (1 - v^2)w + 2v\frac{x + y + z}{\sqrt{3}},$$

or

$$(1 + w)v^2 - 2\frac{x + y + z}{\sqrt{3}}v + 1 - w$$

with solutions

$$v = \frac{\frac{x+y+z}{\sqrt{3}} \pm \sqrt{\frac{(x+y+z)^2}{3} + w^2 - 1}}{1 + w}.$$

When $t = 0$ we have $w = 1$ and $v = 0$ so the correct choice of sign is opposite to the sign of $x + y + z$.

$$\cos(\sqrt{3}t) = \frac{w \mp \frac{x+y+z}{\sqrt{3}}\sqrt{\frac{(x+y+z)^2}{3} + w^2 - 1}}{w^2 + \frac{(x+y+z)^2}{3}},$$

$$\sin(\sqrt{3}t) = \frac{\frac{x+y+z}{\sqrt{3}} \pm w\sqrt{\frac{(x+y+z)^2}{3} + w^2 - 1}}{w^2 + \frac{(x+y+z)^2}{3}}.$$

The other three rows of our matrix equation give

$$q = -\frac{\sin(\sqrt{3}t)}{\sqrt{3}}w + \cos(\sqrt{3}t)x + \frac{\sin(\sqrt{3}t)}{\sqrt{3}}y - \frac{\sin(\sqrt{3}t)}{\sqrt{3}}z,$$

$$r = -\frac{\sin(\sqrt{3}t)}{\sqrt{3}}w - \frac{\sin(\sqrt{3}t)}{\sqrt{3}}x + \cos(\sqrt{3}t)y + \frac{\sin(\sqrt{3}t)}{\sqrt{3}}z,$$

$$s = -\frac{\sin(\sqrt{3}t)}{\sqrt{3}}w + \frac{\sin(\sqrt{3}t)}{\sqrt{3}}x - \frac{\sin(\sqrt{3}t)}{\sqrt{3}}y + \cos(\sqrt{3}t)z.$$

Substituting the expressions found above for $\cos(\sqrt{3}t)$ and $\sin(\sqrt{3}t)$ into these equations gives

$$q = -\frac{wy + wz - xy + xz - y^2 + z^2 \pm \frac{w^2+wy-wz+x^2+xy-xz}{\sqrt{3}}\sqrt{w^2 + \frac{(x+y+z)^2}{3} - 1}}{w^2 + \frac{(x+y+z)^2}{3}},$$

$$r = -\frac{wx + wz + x^2 + xy - yz - z^2 \pm \frac{w^2-wx+wz-xy+y^2+yz}{\sqrt{3}}\sqrt{w^2 + \frac{(x+y+z)^2}{3} - 1}}{w^2 + \frac{(x+y+z)^2}{3}},$$

$$s = -\frac{wy + wz - x^2 - xy + y^2 + yz \pm \frac{w^2+wx-wy+xz-yz+z^2}{\sqrt{3}}\sqrt{w^2 + \frac{(x+y+z)^2}{3} - 1}}{w^2 + \frac{(x+y+z)^2}{3}},$$

where, again, the sign in front of the square root is opposite to that of $x + y + z$. Substituting these into

$$u(w, x, y, z) = f(q, r, s)$$

gives our solution. It's valid where

$$x + y + z \neq 0, \quad w^2 + \frac{(x + y + z)^2}{3} > 1.$$

2. (a) Solve Burgers' Equation

$$\frac{\partial u}{\partial t}(t, x) + u(t, x)\frac{\partial u}{\partial x}(t, x) = 0$$

with initial data

$$u(0, x) = -\frac{x}{\sqrt{1 + x^2}}.$$

You should get a quartic equation for u with coefficients which are polynomials in t and x. You needn't solve this quartic.

(b) For the solution you obtained in the previous part, find $u(t,0)$. For which values of t does uniqueness fail?

Solution:

(a) In general the initial value problem for Burger's Equation with initial data

$$u(0,x) = f(x)$$

was shown in lecture to be given by the implicit equation

$$u = f(x - ut).$$

In this case,

$$u = -\frac{x - ut}{\sqrt{1 + (x - ut)^2}}.$$

Squaring this,

$$u^2 = \frac{(x - ut)^2}{1 + (x - ut)^2},$$

or

$$t^2u^4 - 2txu^3(1 - t^2 + x^2)u^2 + 2txu - x^2 = 0.$$

(b) substuting $x = 0$ gives

$$t^2u^4 + (1 - t^2)u^2 = 0.$$

For $t \leq 1$ the only solution is $u = 0$. For $|t| > 1$ we have the additional solutions

$$u = \pm\sqrt{t^2 - 1}.$$

3. Solve the initial value problem

$$u(0,x) = f(x)$$

for the first order scalar equation

$$\frac{\partial u}{\partial t}(t,x) + \frac{1}{2}\left(\frac{\partial u}{\partial x}(t,x)\right)^2 + \frac{1}{2}x^2 = 0.$$

Note that fully eliminating parameters is not possible with f unspecified. The best you can do it two equations with one extra variable.
Solution:

$$F(t,x,u,u_t,u_x) = u_t + \frac{1}{2}u_x^2 + \frac{1}{2}x^2.$$

We can use[1]

$$H(t, x, u, p_t, p_x, p_u) = -\frac{p_t}{p_u} + \frac{1}{2}\left(-\frac{p_x}{p_u}\right)^2 + \frac{1}{2}x^2.$$

The characteristic equations are then

$$\begin{aligned}
\frac{\partial t}{\partial \tau}(s, \tau) &= -\frac{1}{p_u(s, \tau)}, \\
\frac{\partial x}{\partial \tau}(s, \tau) &= \frac{p_x(s, \tau)}{p_u(s, \tau)^2}, \\
\frac{\partial u}{\partial \tau}(s, \tau) &= \frac{p_t(s, \tau)}{p_u(s, \tau)^2} - \frac{p_x(s, \tau)^2}{p_u(s, \tau)^3}, \\
\frac{\partial p_t}{\partial \tau}(s, \tau) &= 0, \\
\frac{\partial p_x}{\partial \tau}(s, \tau) &= -x(s, \tau), \\
\frac{\partial p_u}{\partial \tau}(s, \tau) &= 0.
\end{aligned}$$

Half of the initial conditions are

$$x = s, \quad t = 0, \quad u = f(s).$$

For the other half we need to figure out the partial derivatives of u initially. Clearly we need

$$-\frac{p_x(s, 0)}{p_u(s, 0)} = \frac{\partial u}{\partial x}(t(s, 0), x(s, 0)) = f'(s).$$

The original partial differential equation then forces

$$-\frac{p_t(s, 0)}{p_u(s, 0)} = \frac{\partial u}{\partial t}(t(s, 0), x(s, 0)) = -\frac{1}{2}f'(s)^2 - \frac{1}{2}s^2.$$

A simple choice is

$$p_t(s, 0) = -\frac{1}{2}f'(s)^2 - \frac{1}{2}s^2, \qquad p_x(s, 0) = f'(s), \quad p_u(s, 0) = -1.$$

[1] We can multiply this by p_u^2 to clear denominators, but that turns out not to simplify things.

The solution of the characteristic inital value problem is then

$$p_u(s,\tau) = -1,$$

$$t(s,\tau) = t,$$

$$p_t(s,\tau) = -\frac{1}{2}f'(s)^2 - \frac{1}{2}s^2,$$

$$x(s,\tau) = s\cos t + f'(s)\sin t,$$

$$p_x(s,\tau) = -s\sin t + f'(s)\cos t,$$

$$u(s,\tau) = f(s) + \frac{1}{2}\left(f'(s)^2 - s^2\right)t.$$

The closest we can get to an explicit solution is the pair of equations

$$x = s\cos t + f'(s)\sin t, \quad u = f(s) + \frac{1}{2}\left(f'(s)^2 - s^2\right)t.$$

relating s, t, u, and x.