1. (a) For each of the following equations or systems give its order and state whether it is linear or non-linear.

   i. The Eikonal Equation

   $$u_t^2 - u_x^2 - u_y^2 - u_z^2 = 0$$

   ii. The Incompressible Euler Equations

   $$\begin{aligned} u_t + uu_x + vu_y + wu_z + p_x &= 0 \\ v_t + uv_x + vv_y + wv_z + p_y &= 0 \\ w_t + uw_x + vw_y + ww_z + p_z &= 0 \\ u_x + v_y + w_z &= 0 \end{aligned}$$

   iii. The Beam Equation

   $$u_{tt} + u_{xxxx} = 0$$

   iv. Burgers' Equation

   $$u_t + uu_x = 0$$

   v. The Hamilton-Jacobi Equation, in the case of a harmonic oscillator

   $$u_t + \frac{1}{2m}u_x^2 + \frac{k}{2}x^2 = 0$$

   (b) To which of the main three equations (Wave, Heat or Laplace) does each of the following apply? Some of these apply to more than one of the equations and some may apply only in certain contexts.

   i. Maximum Principle
   ii. Fundamental solution
   iii. Finite speed of propagation
   iv. Existence and uniqueness of solutions[1]
   v. Mean Value Property

2. (a) State, but do not prove, the global form of energy conservation for the Wave Equation in one space dimension

   $$u_{tt} - c^2 u_{xx} = 0.$$

   (b) Prove the uniqueness of solutions to the initial value problem

   $$u(0, x) = f(x) \quad u_t(0, x) = g(x)$$

   for the Wave Equation in one space dimension. You may use the energy conservation theorem from the previous part if you wish.

[1] under appropriate hypotheses, obviously

3. (a) The explicit solution of the initial value problem

$$u_t - ku_{xx} = 0,$$

$$u(0, x) = f(x)$$

for the Heat Equation on the real line is

$$u(t, x) = \int_{-\infty}^{\infty} (4\pi kt)^{-1/2} \exp\left(-\frac{(x-y)^2}{4kt}\right) f(y)\, dy.$$

Substituting $t = 0$ makes no sense, because of the negative powers both inside and outside the exponential. In what sense, then, does this u satisfy the initial condition $u(0, x) = f(x)$?

(b) State the Maximum Principle for the Heat Equation in both its local and global forms. Be sure to state all hypotheses correctly.

4. (a) What is the solution of the boundary value problem

$$u_{xx} + u_{yy} = 0,$$

$$u(x, 0) = f(x)$$

for the Laplace Equation in the upper halfspace?

(b) Solve for $f(x) = 1/(1 + x^2)$.