MA 3425
Assignment 4
Due 23 November 2012

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1. Solve the Dirichlet problem for the inhomogeneous Wave Equation on the half line:

$$u_{tt} - c^2 u_{xx} = h \quad u(t,0) = 0 \quad u(0,x) = f(x) \quad u_t(0,x) = g(x)$$

for $t, x \geq 0$.
Solution:

For Dirichlet we want the odd extensions of f, g and h:

$$f(-x) = -f(x) \quad g(-x) = -g(x) \quad h(t,-x) = -h(t,x).$$

The solution with this data is given by the usual formula:

$$\begin{aligned} u(t,x) = {} & \frac{1}{2} f(x+ct) + \frac{1}{2} f(x-ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\,dy \\ & + \frac{1}{2c} \int_0^t \int_{x-ct+cs}^{x+ct-cs} h(s,y)\,dy\,ds. \end{aligned}$$

If $x - ct \geq 0$ we are done, since the formula above only involves $f(y)$, $g(y)$ and $h(s,y)$ for $y \geq 0$. If $x - ct < 0$ then we have to write everything in terms of the original, unextended, f, g and h:

$$\begin{aligned} u(t,x) = {} & \frac{1}{2} f(x+ct) - \frac{1}{2} f(ct-x) + \frac{1}{2c} \int_{ct-x}^{x+ct} g(y)\,dy + \frac{1}{2c} \int_0^{x+ct} g(y)\,dy \\ & - \frac{1}{2c} \int_0^{ct-x} g(y)\,dy + \frac{1}{2c} \int_0^{t-x/c} \int_0^{x+ct-cs} h(s,y)\,dy\,ds \\ & - \frac{1}{2c} \int_0^{t-x/c} \int_0^{ct-cs-x} h(s,y)\,dy\,ds + \frac{1}{2c} \int_{t-x/c}^{t} \int_{x-ct+cs}^{x+ct-cs} h(s,y)\,dy\,ds, \end{aligned}$$

or, after a removing integrals which cancel,

$$u(t,x) = \frac{1}{2} f(x+ct) - \frac{1}{2} f(ct-x) + \frac{1}{2c} \int_{ct-x}^{x+ct} g(y)\,dy \\ + \frac{1}{2c} \int_0^{t-x/c} \int_{ct-cs-x}^{x+ct-cs} h(s,y)\,dy\,ds + \frac{1}{2c} \int_{t-x/c}^{t} \int_{x-ct+cs}^{x+ct-cs} h(s,y)\,dy\,ds.$$

2. Prove that

$$\overline{u} = u \quad \overline{x} = \frac{x^2+y^2-1}{x^2+y^2+2y+1} \quad \overline{y} = \frac{2x}{x^2+y^2+2y+1}$$

is a symmetry of the Laplace Equation $u_{xx} + u_{yy} = 0$.
Solution:

To avoid getting hopelessly lost in the algebra, it is best to avoid using the explicit formulae for $\overline{x}$ and $\overline{y}$ until the end of the calculation. This also has the advantage the we can reuse most of the calculation for any other symmetry.

By the chain rule,

$$\frac{\partial}{\partial x} = \frac{\partial \overline{x}}{\partial x} \frac{\partial}{\partial \overline{x}} + \frac{\partial \overline{y}}{\partial x} \frac{\partial}{\partial \overline{y}}.$$

Differentiating twice,

$$\frac{\partial^2}{\partial x^2} = \left(\frac{\partial \overline{x}}{\partial x}\right)^2 \frac{\partial^2}{\partial \overline{x}^2} + 2\left(\frac{\partial \overline{x}}{\partial x} \frac{\partial \overline{y}}{\partial x}\right) \frac{\partial^2}{\partial \overline{x} \partial \overline{y}} + \left(\frac{\partial \overline{y}}{\partial x}\right)^2 \frac{\partial^2}{\partial \overline{y}^2} + \frac{\partial^2 \overline{x}}{\partial x^2} \frac{\partial}{\partial \overline{x}} + \frac{\partial^2 \overline{y}}{\partial x^2} \frac{\partial}{\partial \overline{y}}.$$

Similarly,

$$\frac{\partial^2}{\partial y^2} = \left(\frac{\partial \overline{x}}{\partial y}\right)^2 \frac{\partial^2}{\partial \overline{x}^2} + 2\left(\frac{\partial \overline{x}}{\partial y} \frac{\partial \overline{y}}{\partial y}\right) \frac{\partial^2}{\partial \overline{x} \partial \overline{y}} + \left(\frac{\partial \overline{y}}{\partial y}\right)^2 \frac{\partial^2}{\partial \overline{y}^2} + \frac{\partial^2 \overline{x}}{\partial y^2} \frac{\partial}{\partial \overline{x}} + \frac{\partial^2 \overline{y}}{\partial y^2} \frac{\partial}{\partial \overline{y}}.$$

So,

$$\overline{u}_{\overline{x}\overline{x}} + \overline{u}_{\overline{y}\overline{y}} = a u_{xx} + b u_{xy} + c u_{yy} + d u_x + e u_y,$$

where

$$a = \left(\frac{\partial \overline{x}}{\partial y}\right)^2 + \left(\frac{\partial \overline{x}}{\partial y}\right)^2$$
$$b = 2\left(\frac{\partial \overline{x}}{\partial x} \frac{\partial \overline{y}}{\partial x} + \frac{\partial \overline{x}}{\partial y} \frac{\partial \overline{y}}{\partial y}\right)$$
$$c = \left(\frac{\partial \overline{y}}{\partial x}\right)^2 + \left(\frac{\partial \overline{y}}{\partial y}\right)^2$$
$$d = \frac{\partial^2 \overline{x}}{\partial x^2} + \frac{\partial^2 \overline{x}}{\partial y^2}$$
$$e = \frac{\partial^2 \overline{y}}{\partial x^2} + \frac{\partial^2 \overline{x}}{\partial y^2}$$

Now we just compute derivatives.

$$\frac{\partial \overline{x}}{\partial x} = -\frac{\partial \overline{y}}{\partial y} = \frac{4xy + 4x}{(x^2 + y^2 + 2y + 1)^2},$$

$$\frac{\partial \overline{x}}{\partial y} = \frac{\partial \overline{y}}{\partial x} = \frac{-2x^2 + 2y^2 + 4y + 2}{(x^2 + y^2 + 2y + 1)^2},$$

$$\frac{\partial^2 \overline{x}}{\partial x^2} = -\frac{\partial^2 \overline{x}}{\partial y^2} = \frac{-12x^2y + 4y^3 - 12x^2 + 12y^2 + 12y + 4}{(x^2 + y^2 + 2y + 1)^3},$$

$$\frac{\partial^2 \overline{y}}{\partial x^2} = -\frac{\partial^2 \overline{y}}{\partial y^2} = \frac{4x^3 - 12xy^2 - 24xy - 12x}{(x^2 + y^2 + 2y + 1)^3}.$$

Then $b = d = e = 0$ and

$$a = c = \frac{4}{(x^2 + y^2 + 2y + 1)^2}.$$

So

$$\overline{u}_{\overline{x}\overline{x}} + \overline{u}_{\overline{y}\overline{y}} = 4\frac{u_{xx} + u_{yy}}{(x^2 + y^2 + 2y + 1)^2}$$

and hence $\overline{u}_{\overline{x}\overline{x}} + \overline{u}_{\overline{y}\overline{y}} = 0$ if and only if $u_{xx} + u_{yy} = 0$.