MA 3425
Assignment 3
Due 30 October 2012

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1. Show that for solutions of the homogeneous Heat Equation, $u_t - ku_{xx} = 0$, on an interval $[a, b]$ with Neumann boundary conditions at the endpoints, $u_x(t, a) = u_x(t, b) = 0$, that the integral
$$\int_a^b u(t,x)\,dx$$
is constant.
Hint: Differentiate under the integral sign and integrate by parts.
Solution: Differentiating under the integral sign,
$$\frac{d}{dt}\int_a^b u(t,x)\,dx = \int_a^b u_t(t,x)\,dx = k\int_a^b u_{xx}(t,x)\,dx.$$
The differentiation under the integral sign is justified because u_t is continuous, and hence integrable, on the finite interval $[a, b]$. By the Fundamental Theorem of Calculus then,
$$\frac{d}{dt}\int_a^b u(t,x)\,dx = ku_x(t,b) - ku_x(t,a).$$
The right hand side is zero if u satisfies the Neumann condition at both endpoints.

2. Show, by means of an example, that the integral from the preceding problem need not be constant if Dirichlet boundary conditions, $u(t, a) = u(t, b) = 0$, are imposed instead of Neumann conditions. *Hint:* You have seen a counterexample already.
Solution: One counterexample is
$$u(t,x) = \exp(-k\omega^2 t)\cos(\omega t)$$

on the interval $\left[-\frac{\pi}{2\omega}, \frac{\pi}{2\omega}\right]$. This satisfies the Dirichlet condition because $\cos\left(\pm\frac{\pi}{2}\right) = 0$. Integrating,

$$\int_{-\frac{\pi}{2\omega}}^{\frac{\pi}{2\omega}} u(t,x)\,dx = \frac{2}{\omega}\exp(-k\omega^2 t),$$

which is not constant if $\omega \neq 0$.

3. Suppose that u satisfies the initial value problem $u(0,x) = f(x)$ for the homogeneous Heat Equation on the real line. Suppose further that f, in addition to being bounded and continuous, is such that $|f(x)|^2$ is integrable. Show that

$$\sup_{-\infty<x<\infty} |u(t,x)|,$$

which we know by the maximum principle is a bounded function of t, tends to zero as t tends to infinity.
Hint: You need an inequality for integrals. Several of these were discussed in lecture, and in a paper linked from the course web site.
Solution: We have the integral representation

$$u(t,x) = \pi^{-1/2}\int_{-\infty}^{\infty} \exp(-z^2) f(x + z\sqrt{4kt})\,dz$$

By Cauchy-Schwarz,

$$|u(t,x)| \leq \pi^{-1/2} I^{1/2}\left(\int_{-\infty}^{\infty} |f(x+z\sqrt{4kt})|^2\,dz\right)^{1/2},$$

where $I = \int_{-\infty}^{\infty}\exp(-2z^2)\,dz < \infty$. Making the change of variable $z = \frac{y-x}{\sqrt{4kt}}$,

$$\int_{-\infty}^{\infty} |f(x+z\sqrt{4kt})|^2\,dz = (4kt)^{-1/2}\int_{-\infty}^{\infty} |f(y)|^2\,dy.$$

It follows that

$$\sup_{-\infty<x<\infty} |u(t,x)| \leq (4\pi kt/I)^{-1/2}$$

and hence that

$$\lim_{t\to\infty}\sup_{-\infty<x<\infty} |u(t,x)| = 0.$$

The exact value of I is irrelevant to the problem, but a simple change of variable shows that

$$I = \sqrt{\frac{\pi}{2}}.$$