MA 3425
Assignment 2
Due 16 October 2012

`Id: 3425-f2012-2.m4,v 1.3 2012/10/12 12:11:10 john Exp john`

1. For which p, q, r is the scaling $\overline{u} = \lambda^p u$, $\overline{t} = \lambda^q t$, $\overline{x} = \lambda^r x$ a symmetry of the cubic non-linear wave equation

$$u_{tt} - c^2 u_{xx} + u^3 = 0?$$

Solution: From the chain rule,

$$\overline{u}_{\overline{tt}} - c^2 \overline{u}_{\overline{xx}} + \overline{u}^3 = \lambda^{p-2q} u_{tt} - c^2 \lambda^{p-2r} u_{xx} + \lambda^{3p} u^3.$$

To ensure that this is zero if and only if $u_{tt} - c^2 u_{xx} + u^3$ is, we need it to be a non-zero multiple of $u_{tt} - c^2 u_{xx} + u^3$. This happens if and only if $q = r = -p$.

2. Show that for solutions of the cubic non-linear wave equation the energy

$$\int_{-\infty}^{\infty} \epsilon(t, x)\, dx$$

with energy density

$$\frac{1}{2} u_t^2 + \frac{c^2}{2} u_x^2 + \frac{1}{4} u^4$$

is constant.
Hint: Of the proofs of energy conservation for the ordinary wave equation presented in class, only the last one, the one with the trapezoids, can be adapted to this problem.
Solution: The differential form of energy conservation is again

$$\epsilon_t = \mu_x + (u_{tt} - c^2 u_{xx} + u^3) u_t = \mu_x$$

for solutions. Here

$$\mu = c^2 u_t u_x,$$

as before. Integrating over the trapezoid defined by

$$0 \le t \le T, \quad x - ct \ge a - cT, \quad x + ct \ge b + cT$$

gives, as in class,

$$\begin{aligned}\int_a^b \epsilon(T,x)\,dx = &\int_{a-cT}^{b+cT} \epsilon(0,x)\,dx \\ &- \int_0^T (c\epsilon + \mu)(t, a - cT + ct)\,dt \\ &- \int_0^T (c\epsilon - \mu)(t, b + cT - ct)\,dt.\end{aligned}$$

Now,

$$c\epsilon + \mu = \frac{c}{2}(u_t + cu_x)^2 + \frac{c}{4}u^4$$

and

$$c\epsilon - \mu = \frac{c}{2}(u_t - cu_x)^2 + \frac{c}{4}u^4.$$

The second summands on the right are new, but they don't change the fact that the integrands are everywhere non-negative. It follows then that

$$\int_a^b \epsilon(T,x)\,dx \le \int_{a-cT}^{b+cT} \epsilon(0,x)\,dx.$$

Similarly, integration of $\epsilon_t = \mu_x$ over the trapezoid

$$0 \le t \le T, \quad x + ct \ge a + cT, \quad x - ct \ge b - cT$$

gives

$$\begin{aligned}\int_a^b \epsilon(T,x)\,dx = &\int_{a+cT}^{b-cT} \epsilon(0,x)\,dx \\ &+ \int_0^T (c\epsilon - \mu)(t, a + cT - ct)\,dt \\ &+ \int_0^T (c\epsilon + \mu)(t, b - cT + ct)\,dt,\end{aligned}$$

so

$$\int_a^b \epsilon(T,x)\,dx \ge \int_{a+cT}^{b-cT} \epsilon(0,x)\,dx.$$

From the squeeze principle for limits it follows that

$$\int_{-\infty}^{\infty} \epsilon(T,x)\,dx$$

converges if

$$\int_{-\infty}^{\infty} \epsilon(0,x)\,dx$$

does and that the integrals are equal.