MA 3425
Assignment 1
Due 2 October 2012

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1. For each of the following, say whether it is a scalar equation or system, give the order, and state whether it is linear or non-linear.

   (a) The 1+3 Dimensional Klein-Gordon Equation

   $$u_{tt} - u_{xx} - u_{yy} - u_{zz} + u = 0$$

   Solution: Second order linear scalar.

   (b) The Incompressible Euler Equations

   $$\begin{aligned} u_t + uu_x + vu_y + wu_z + p_x &= 0 \\ v_t + uv_x + vv_y + wv_z + p_y &= 0 \\ w_t + uw_x + vw_y + ww_z + p_z &= 0 \\ u_x + v_y + w_z &= 0 \end{aligned}$$

   Solution: First order non-linear system.

   (c) The Korteweg-de Vries Equation

   $$u_t - u_{xxx} + 6uu_x = 0$$

   Solution: Third order nonlinear scalar.

2. $u(t, x) = \cos(ckt)\cos(kx)$ is called a standing wave.

   (a) Verify that it satisfies the homogeneous Wave Equation

   $$u_{tt} - c^2 u_{xx} = 0$$

   for any k.
   Solution: Differentiate. Twice.

(b) Write it in the form $u(t,x) = \varphi(x+ct) + \psi(x-ct)$.
Solution: $\cos(ckt)\cos(kx) = \frac{1}{2}\cos(ckt+kx) + \frac{1}{2}\cos(ckt-kx)$.

3. Solve the initial value problem $u(0,x) = f(x)$, $u_t(0,x) = g(x)$ for the Wave Equation,

(a) with $f(x) = (1+x^2)^{-1}$ and $g(x) = 0$,
Solution: Substituting into the solution formula

$$u(t,x) = \frac{1}{2}f(x+ct) + \frac{1}{2}f(x-ct) + \frac{1}{2}\int_{x-ct}^{x+ct} g(y)\,dy$$

gives

$$u(t,x) = \frac{1}{2}\left(\frac{1}{1+x^2+2ctx+c^2t^2} + \frac{1}{1+x^2-2ctx+c^2t^2}\right).$$

This can be "simplified" to

$$u(t,x) = \frac{1+x^2+c^2t^2}{(1+x^2+c^2t^2)^2+(2ctx)^2}$$

if desired.

(b) with $f(x) = 0$ and $g(x) = (1+x^2)^{-1}$.
Solution: Substituting into the solution formula gives

$$u(t,x) = \frac{1}{2c}\left(\arctan(x+ct) - \arctan(x-ct)\right).$$

This can be "simplified" to

$$u(t,x) = \frac{1}{2c}\arctan\frac{2ct}{1-x^2+c^2t^2}.$$