MAU34215 Assignment 4
Due 26 November 2025
Solutions

1. Prove that the following are symmetries of Burgers' equation.

(a)

$$\tilde{u}(t,x) = u(t, x - vt) + v$$

for all v.

Solution: Taking derivatives using the chain rule,

$$\frac{\partial \tilde{u}}{\partial t}(t,x) = \frac{\partial u}{\partial t}(t, x - vt) - v\frac{\partial u}{\partial x}(t, x - vt)$$

and

$$\frac{\partial \tilde{u}}{\partial x}(t,x) = \frac{\partial u}{\partial x}(t, x - vt)$$

so

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial t}(t,x) + \tilde{u}(t,x)\frac{\partial \tilde{u}}{\partial x}(t,x) &= \frac{\partial u}{\partial t}(t, x - vt) + \left(\tilde{u}(t,x) - v\right)\frac{\partial u}{\partial x}(t, x - vt) \\
&= \frac{\partial u}{\partial t}(t, x - vt) + u(t, x - vt)\frac{\partial u}{\partial x}(t, x - vt)
\end{aligned}$$

and the left hand side is equal to zero if and only if the right hand side is.

(b)

$$\tilde{u}(t,x) = u(-t,-x).$$

Solution:

$$\frac{\partial \tilde{u}}{\partial t}(t,x) = -\frac{\partial u}{\partial t}(-t,-x)$$

and

$$\frac{\partial \tilde{u}}{\partial x}(t,x) = -\frac{\partial u}{\partial x}(-t,-x)$$

so

$$\begin{aligned}
\frac{\partial \tilde{u}}{\partial t}(t,x) + \tilde{u}(t,x)\frac{\partial \tilde{u}}{\partial x}(t,x) &= -\frac{\partial u}{\partial t}(-t,-x) - \tilde{u}(t,x)\frac{\partial u}{\partial x}(-t,-x) \\
&= -\left[\frac{\partial u}{\partial t}(-t,-x) + u(-t,-x)\frac{\partial u}{\partial x}(-t,-x)\right].
\end{aligned}$$

(c)

$$u(t,x) = \mu u(t/\mu, x/\mu^2)$$

for all non-zero μ.

Solution:

$$\frac{\partial \tilde{u}}{\partial t}(t,x) = \frac{\partial u}{\partial t}(t/\mu, x/\mu^2)$$

and

$$\frac{\partial \tilde{u}}{\partial x}(t,x) = \frac{1}{\mu}\frac{\partial u}{\partial x}(t/\mu, x/\mu^2)$$

so

$$\begin{aligned}\frac{\partial \tilde{u}}{\partial t}(t,x) + \tilde{u}(t,x)\frac{\partial \tilde{u}}{\partial x}(t,x) &= \frac{\partial u}{\partial t}(t/\mu, x/\mu^2) + \frac{\tilde{u}(t,x)}{\mu}\frac{\partial u}{\partial x}(t/\mu, x/\mu^2)\\ &= \frac{\partial u}{\partial t}(t/\mu, x/\mu^2) + \tilde{u}(t/\mu, x\mu^2)\frac{\partial u}{\partial x}(t/\mu, x/\mu^2).\end{aligned}$$

2. We proved existence and uniqueness for the Dirichlet problem for the unit disc, but not for the Neumann problem. The Neumann problem is, given a continuous function g on the unit circle, to find a continuously differentiable function u on closed unit disc which is twice continuously differentiable in the open unit disc and satisfies the Laplace equation there, and whose radial derivative,

$$\frac{x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial y},$$

is equal to g on the unit circle.

   (a) Show that the Neumann problem does not have unique solutions by giving a function g and two distinct solutions, u_1 and u_2 to the Neumann problem for this g.

   Hint: There are a lot of possible choices but some of them are extremely simple, so just look for very simple solutions of the Laplace equation and see whether any work.

   Solution: If we take $g = 0$ then any constant function will solve the Neumann problem, so we can just take $u_j = j$, for example.

   (b) Show that there are no solutions to the Neumann problem unless

$$\int_{-\pi}^{\pi} g(\cos\theta, \sin\theta)\, d\theta = 0.$$

Hint: Apply Green's theorem to the functions

$$p(x, y) = \frac{\partial u}{\partial y}, \quad q(x, y) = -\frac{\partial u}{\partial x}$$

in an appropriate region.

Solution: The following almost works. Take the region R from the theorem to be the closed unit disc, in which case the boundary consists of a single curve, the unit circle, which we can parameterise in the usual way by an angle. Then

$$\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} = -\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} = 0$$

so Green's theorem gives

$$\int_C (p\, dx + q\, dy) = 0.$$

With the usual paramaterisation of the circle this integral is

$$\int_{-\pi}^{\pi} \frac{\partial u}{\partial y}(\cos\theta, \sin\theta)\cos\theta + \frac{\partial u}{\partial x}(\cos\theta, \sin\theta)\sin\theta$$

which is just

$$\int_{-\pi}^{\pi} g(\cos\theta, \sin\theta)\, d\theta.$$

There is one problem with the argument above. Green's Theorem requires p and q to be continuously differentiable in the region R and our p and q are only known to be continuous there and differentiable in the interior. To get around this we can apply Green's Theorem to the disc of radius r where $r < 1$ and then take limits as r tends to 1 from below. For exchanging the limit and integral we only need continuity, so that's fine.

3. The region

$$R = \left\{(x, y) \in \mathbf{R}^2 \colon x \geq 0, y \geq 0, x^2 + y^2 \leq 1\right\}$$

in the plane has a boundary consisting of the three curves

$$C_1 = \left\{(x, y) \in \mathbf{R}^2 \colon x \geq 0, y = 0, x^2 + y^2 \leq 1\right\},$$

$$C_2 = \left\{(x,y) \in \mathbf{R}^2 \colon x \geq 0, y \geq 0, x^2+y^2 = 1\right\}$$

and

$$C_3 = \left\{(x,y) \in \mathbf{R}^2 \colon x = 0, y \geq 0, x^2+y^2 \leq 1\right\}.$$

There is a transformation of the plane which is a symmetry of the Laplace equation and maps R to itself in such a way that C_1 is mapped to C_2, C_2 is mapped to C_3 and C_3 is mapped to C_1. The goal of this problem is to find that transformation.

(a) As described in the notes, Lorentz matrices correspond to symmetries of Laplace. The symmetry we're looking for will map the x axis to the unit circle, the unit circle to the y axis and the y axis to the x axis. What conditions do we need on our Lorentz matrix to accomplish this.

Solution: It's simplest to use the technique described in lecture. If vectors $\mathbf{v}$ such that $\mathbf{v}^T G \mathbf{v} > 0$ are associated to a circle $C_{\mathbf{v}}$ by

$$C_{\mathbf{v}} = \{(x,y) \in \mathbf{R}^2 \colon \mathbf{v}^T j(x,y) = 0\}$$

then the image $F_A(C_{\mathbf{v}})$ of $C_{\mathbf{v}}$ under the mapping F_A of the plane associated with the Lorentz matrix A is $C_{A^{-T}\mathbf{v}}$. The y axis, x axis and unit circle are the curves associated to the first three standard basis vectors $(1,0,0,0)$, $(0,1,0,0)$, and $(0,0,1,0)$.

The transformation we're looking for must permute the y axis, x axis and unit circle. the condition that F_A maps C_1 to C_2, for example, implies that

$$A^{-T}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

for some non-zero number λ. The appearance of this multiple λ in the equation reflects the fact that $C_{\lambda\mathbf{v}}$ is the same circle or line as $C_{\mathbf{v}}$. Now A is a Lorentz matrix so A^{-T} is also a Lorentz matrix, so we must have $\lambda = \pm 1$. There are a few different ways to determine the sign. One is trial and error. We will have an undetermined sign in each of the three conditions so we can try each of the eight possible combinations. A better option is to looks at inequalities. The circle or line $C_{\mathbf{v}}$ divides the plane into regions

$$P_{\mathbf{v}} = \{(x,y) \in \mathbf{R}^2 \colon \mathbf{v}^T j(x,y) > 0\}$$

and

$$N_{\mathbf{v}} = \{(x,y) \in \mathbf{R}^2 \colon \mathbf{v}^T j(x,y) < 0\}$$

where $\mathbf{v}^T j(x,y)$ is positive or negative, respectively. Then then the image $F_A(P_{\mathbf{v}})$ of $P_{\mathbf{v}}$ under the mapping F_A of the plane associated with the Lorentz matrix A is $P_{A^{-T}\mathbf{v}}$ and the image $F_A(N_{\mathbf{v}})$ of $N_{\mathbf{v}}$ is $N_{A^{-T}\mathbf{v}}$, provided the last entry of A, a_{44} is positive, which we can always arrange since A and $-A$ give the same transformation of the plane. Since the transformation we're looking for takes the upper half plane to interior of the unit circle we must have $\lambda = +1$, so

$$A^{-T} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

In a similar way we get the conditions

$$A^{-T} \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

and

$$A^{-T} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

from the condition that C_2 is mapped to C_3 and C_3 is mapped to C_1.

(b) Find a Lorentz matrix with the required properties.

Solution: From the three conditions above it follows that A^{-T} has the form

$$A^{-T} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ ? & ? & ? & ? \end{bmatrix}.$$

Since A^{-T} is a Lorentz matrix the first three entries in the last row are all 0 and the last one is ± 1, but we've already chosen to make the last one positive, so

$$A^{-T} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

It happens that for this matrix

$$A = A^{-T}.$$

(c) What is the corresponding symmetry of the Laplace equation?
Solution: The symmetry is

$$\tilde{u}(x, y) = (u \circ F_{A^{-1}})(x, y) = u(p(A^{-1} j(x, y)))$$

In this case

$$A^{-1} = A^T = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

so

$$\begin{aligned} \tilde{u}(x, y) &= u\left(p\left(A^{-1}\begin{bmatrix} 2x \\ 2y \\ 1 - x^2 - y^2 \\ 1 + x^2 + y^2 \end{bmatrix}\right)\right) = u\left(p\left(\begin{bmatrix} 2y \\ 1 - x^2 - y^2 \\ 2x \\ 1 + x^2 + y^2 \end{bmatrix}\right)\right) \\ &= u\left(\frac{2y}{(x+1)^2 + y^2}, \frac{1 - x^2 - y^2}{(x+1)^2 + y^2}\right). \end{aligned}$$

The appearance of A^{-1} rather than A is necessary if we want composition of symmetries to correspond to multiplication of matrices in the correct order, as explained in lecture. If you took A instead you would get

$$\tilde{u}(x, y) = u\left(\frac{1 - x^2 - y^2}{x^2 + (y+1)^2}, \frac{2x}{x^2 + (y+1)^2}\right)$$

instead, which is also a symmetry, just not the one which was asked for.