MAU34215 Assignment 3
Due 5 November 2025
Solutions

1. In the notes and in lecture we mostly restricted our attention to bounded solutions and bounded initial data. There were good reasons for this but unbounded data and solutions do occur in applications. It's still possible to draw some useful conclusions provided the data aren't too badly unbounded.

   (a) Find solutions to the usual initial value problem

   $$\frac{\partial u}{\partial t}(t,x) - k\frac{\partial^2 u}{\partial x^2}(t,x) = 0$$

   for $t \geq 0$ and $x \in \mathbf{R}$ with

   $$u(0,x) = f(x)$$

   with the following initial data:

   i. $$f(x) = \exp(rx),$$

   ii. $$f(x) = \exp(-rx),$$

   iii. $$f(x) = \cosh(rx).$$

   Here r is a real number.

   Note: If we exclude the rather uninteresting case of $r = 0$ then these f's are unbounded, so few of the the theorems from lecture or the notes apply. Depending on how you solve the problem you may therefore need to check some things which you could otherwise use those theorems for.

   Solution: The most straightforward option is to use the explicit solution formula

   $$u(t,x) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-y)^2}{4kt}\right) f(y)\, dy.$$

For the first datum this gives

$$
\begin{aligned}
u(t,x) &= \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-y)^2}{4kt}\right) \exp(ry)\, dy \\
&= \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-y)^2 - 4krty}{4kt}\right)\, dy \\
&= \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-y+2krt)^2 - 4krtx - 4k^2r^2t^2}{4kt}\right)\, dy \\
&= \exp(kr^2t + rx) \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-y+2krt)^2}{4kt}\right)\, dy.
\end{aligned}
$$

Now by a simple linear change of variable

$$
\frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-y+2krt)^2}{4kt}\right)\, dy = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{z^2}{4kt}\right)\, dz = 1
$$

so

$$
u(t,x) = \exp(kr^2t + rx).
$$

The explicit solution formula was only proved for bounded f so we don't actually know that this is a solution but it clearly satisfies the initial conditions and an easy calculation gives

$$
\frac{\partial u}{\partial t}(t,x) = kr^2 \exp(kr^2t + rx),
$$

$$
\frac{\partial u}{\partial x}(t,x) = r \exp(kr^2t + rx),
$$

and

$$
\frac{\partial^2 u}{\partial x^2}(t,x) = r^2 \exp(kr^2t + rx),
$$

so

$$
\frac{\partial u}{\partial t}(t,x) - k\frac{\partial^2 u}{\partial x^2}(t,x) = 0.
$$

This solves the first problem. There is a better way to solve this problem though. One of the few things we did with the diffusion equation before imposing boundedness assumptions was to find symmetries, one of which was

$$
(G_v u)(t,x) = \exp\left(\frac{v^2 t}{4k} - \frac{vx}{2k}\right) u(t, x - vt).
$$

Taking

$$v = -2kr$$

and the trivial solution

$$u(t,x) = 1$$

we get that

$$\exp(kr^2t + rx)$$

is a solution and it clearly satisfies the first initial condition. While we could solve second one the same way it's simpler to note that no assumptions were made on r beyond the fact that it's real so we can just substitute $-r$ for r, which gives

$$u(t,x) = \exp(kr^2t - rx).$$

For the third we can use linearity to say that if the two functions above are solutions to the diffusion equation then so is their average

$$\begin{aligned} u(t,x) &= \frac{\exp(kr^2t + rx) + \exp(kr^2t - rx)}{2} \\ &= \exp(kr^2t)\frac{\exp(rx) + \exp(-rx)}{2} = \exp(kr^2t)\cosh(rx). \end{aligned}$$

(b) Show that if

$$|f(x)| \leq C\exp(rx)$$

or

$$|f(x)| \leq C\exp(-rx)$$

for some real C and r then the initial value problem has a unique solution u satisfying

$$|u(t,x)| \leq C\exp(kr^2t + rx)$$

or

$$|u(t,x)| \leq C\exp(kr^2t - rx),$$

respectively, for all $t \geq 0$ and $x \in \mathbf{R}$.

Hint: It's certainly possible to follow the existence and uniqueness proofs from the notes making various changes to fit these new hypotheses and conclusions but those proofs were long and complicated. Can you use symmetries instead?

Solution: Because of the spatial reflection symmetry of the diffusion equation, or just the fact that we can substitute $-r$ for r, it suffices to consider only the case

$$|f(x)| \leq C \exp(rx).$$

We use the symmetry

$$(G_v u)(t,x) = \exp\left(\frac{v^2 t}{4k} - \frac{vx}{2k}\right) u(t, x - vt)$$

from the notes. Define $\tilde{f}$ by

$$\tilde{f}(x) = \exp(-rx) f(x).$$

Then $\tilde{f}$ is bounded. In fact

$$|\tilde{f}(x)| \leq C$$

for all x. It follows that there is a unique bounded solution $\tilde{u}$ to the diffusion equation with

$$\tilde{u}(0,x) = \tilde{f}(x).$$

In fact we saw that this solution satisfies the same bounds as $\tilde{f}$, so

$$|\tilde{u}(t,x)| \leq C$$

for all $t \geq 0$ and $x \in \mathbf{R}$. Let $u = G_{-2kr}\tilde{u}$. In other words,

$$u(t,x) = \exp(kr^2 t + rx)\tilde{u}(t, x + 2krt)$$

Then u satisfies the diffusion equation and

$$u(0,x) = \exp(rx)\tilde{u}(0,x) = \exp(rx)\tilde{f}(x) = f(x).$$

Also

$$|u(t,x)| = \exp(kr^2 t + rx)|\tilde{u}(t, x + 2krt)| \leq C \exp(kr^2 t + rx).$$

The above argument gives existence, but for uniqueness we need one additional step. For any solution of the initial value problem for u which satisfies the bound the function $G_{2kr}u$ solves the initial value problem as $\tilde{u}$ and we already have uniqueness for this problem so $G_{2kr}u = \tilde{u}$.

(c) Show that if

$$|f(x)| \leq C \cosh(rx)$$

for some real C and r then the initial value problem has a solution u satisfying

$$|u(t,x)| \leq C \exp(kr^2 t) \cosh(rx).$$

for all $t \geq 0$ and $x \in \mathbf{R}$.

Hint: Uniqueness is still true with these hypotheses but I haven't asked you prove it, only existence. Again, it's possible, but unnecessary, to follow the existence proof from the notes. Can you use linearity instead?

Solution: Write

$$f(x) = f_+(x) + f_-(x)$$

where

$$f_+(x) = 2\frac{\exp(rx)}{\exp(rx) + \exp(-rx)}$$

and

$$f_-(x) = 2\frac{\exp(-rx)}{\exp(rx) + \exp(-rx)}$$

Then

$$f(x) = \frac{1}{2} f_+(x) + \frac{1}{2} f_-(x),$$
$$|f_+(x)| \leq C \exp(rx),$$

and

$$|f_-(x)| \leq C \exp(-rx).$$

From the previous part of this problem we know that there are solutions u_+ and u_- to the diffusion equation satisfying the initial conditions

$$u_+(0,x) = f_+(x)$$

and

$$u_-(0,x) = f_-(x)$$

and the bounds

$$|u_+(t,x)| \leq C \exp(kr^2 t + rx)$$

and

$$|u_-(t,x)| \le C\exp(kr^2t - rx).$$

Let

$$u = \frac{1}{2}u_+ + \frac{1}{2}u_-.$$

By linearity we know that u satisfies the diffusion equation. It also satisfies the initial condition $u(0,x) = f(x)$ and the bound

$$|u(t,x)| \le C\exp(kr^2t)\cosh(rx).$$

2. (a) Suppose u is a solution to the initial value problem

$$\frac{\partial u}{\partial t}(t,x) - k\frac{\partial^2 u}{\partial x^2}(t,x) = 0, \quad u(0,x) = f(x)$$

where f is bounded and continuous. Prove that

$$\int_{-\infty}^{+\infty} |u(t,x)|\,dx \le \int_{-\infty}^{+\infty} |f(x)|\,dx.$$

Solution: The hypotheses of our uniqueness theorem are satisfied and we know that in that case the unique solution is given by the explicit formula

$$u(t,x) = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{+\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right) f(y)\,dy$$

so

$$|u(t,x)| = \left|\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{+\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right) f(y)\,dy\right|$$

and

$$\int_{-\infty}^{+\infty} |u(t,x)|\,dx = \int_{-\infty}^{+\infty} \left|\frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{+\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right) f(y)\,dy\right|\,dx.$$

The absolute value of an integral is always less than or equal to the absolute value of the integral so

$$\int_{-\infty}^{+\infty} |u(t,x)|\,dx \le \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{+\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right) |f(y)|\,dy\,dx.$$

The factor $1/\sqrt{4\pi kt}$ is constant and so can be brought inside the inner integral. By Fubini we can then exchange the integrals, so

$$\int_{-\infty}^{+\infty} |u(t,x)|\,dx \le \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-y)^2}{4kt}\right) |f(y)|\,dx\,dy.$$

The factor $|f(y)|$ is a constant as far as the inner integral is concerned, so we can bring it outside the inner integral, obtaining

$$\begin{aligned}\int_{-\infty}^{+\infty} |u(t,x)|\,dx &\le \int_{-\infty}^{+\infty} |f(y)| \int_{-\infty}^{+\infty} \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{(x-y)^2}{4kt}\right) dx\,dy \\ &\le \int_{-\infty}^{+\infty} |f(y)|\,dy.\end{aligned}$$

This is what we were meant to prove, except for the name of the variable of integration on the right hand side, which is irrelevant.

(b) With hypotheses as above prove that the inequality above is strict if and only if there are points x_1 and x_2 where $f(x_1) > 0$ and $f(x_2) < 0$.

Solution: If there are no such points then either $f(x) \ge 0$ for all x or $f(x) \le 0$ for all x. In the former case our positivity theorem guarantees that $u(t,x) \ge 0$ for all $t \ge 0$ and all x. But then

$$\int_{-\infty}^{+\infty} |u(t,x)|\,dx = \int_{-\infty}^{+\infty} u(t,x)\,dx$$

and

$$\int_{-\infty}^{+\infty} |f(x)|\,dx \int_{-\infty}^{+\infty} f(x)\,dx.$$

and we already know that

$$\int_{-\infty}^{+\infty} u(t,x)\,dx = \int_{-\infty}^{+\infty} f(x)\,dx.$$

so

$$\int_{-\infty}^{+\infty} |u(t,x)|\,dx = \int_{-\infty}^{+\infty} u(t,x)\,dx.$$

The argument for the case where $f(x) \le 0$ for all x is the same, except that we use $-u$ in place of u. Thus the inequality above is strict only if there are x_1 and x_2 with $f(x_1) > 0$ and $f(x_2) < 0$.

To prove the converse we note that if there are x_1 and x_2 with $f(x_1) > 0$ and $f(x_2) < 0$ then the integrand in

$$\int_{-\infty}^{+\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right) f(y)\, dy\, dx$$

is continuous and changes sign so the absolute value of the integral is strictly less than the integral of the absolute value, and so

$$|u(t,x)| < \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right) f(y)\, dy\, dx.$$

We can therefore run the argument from the previous part with strict inequalities in place of non-strict ones to obtain

$$\int_{-\infty}^{+\infty} |u(t,x)|\, dx < \int_{-\infty}^{+\infty} |f(x)|\, dx.$$

3. Solve Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

with initial data

$$u(0,x) = x + \sqrt{1+x^2}.$$

Solution: For general initial data f the solution to

$$u(0,x) = f(x)$$

is given by solving

$$u = f(x - ut).$$

In this case

$$f(x) = x + \sqrt{1+x^2}$$

so a bit of algebra gives the equations

$$u = x - ut + \sqrt{1 + (x-ut)^2},$$

$$(1+t)u - x = \sqrt{1 + x^2 - 2utx + u^2t^2},$$

$$(1+2t+t^2)u^2 - 2(1+t)xu + x^2 = 1 + x^2 - 2utx + u^2t^2,$$

$$(1+2t)u^2 - 2xu - 1 = 0,$$

and

$$u = \frac{2x \pm \sqrt{4x^2 + 4(1+2t)}}{2(1+2t)} = \frac{x \pm \sqrt{1+2t+x^2}}{1+2t}.$$

In order to satisfy the initial conditions we need to take the + sign so

$$u(t,x) = \frac{x + \sqrt{1+2t+x^2}}{1+2t}.$$