MAU34215 Assignment 2
Due 15 October 2025
Solutions

1. Consider the initial value problem

$$u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = g(x)$$

for the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

in $\mathbf{R} \times [0, +\infty)$ with Dirichlet boundary conditions at $x = 0$, i.e.

$$u(t, 0) = 0.$$

(a) What conditions on f and g are needed to obtain a classical solution.

Solution: g should be continuously differentiable and f should be twice continuously differentiable. Also $f(0) = 0$, $g(0) = 0$ and $f''(x) = 0$.

(b) For f and g satisfying the conditions you gave in the first part prove that there is at least one classical solution.

Solution: We solve the initial value problem in all of $\mathbf{R}^2$ with initial data which are odd extensions of f and g, so

$$f(x) = -f(-x), \quad g(x) = -g(-x)$$

for $x < 0$. The extended f and g are twice continuously differentiable and once continuously differentiable, respectively, so this initial value problem has a classical solution u. The functions u and $\tilde{u}$, where

$$\tilde{u}(t,x) = -u(t,-x)$$

satisfy the same initial conditions and so, by the uniqueness theorem for solutions in $\mathbf{R}^2$, are the same function. In other words,

$$u(t,x) = -u(t,-x)$$

Taking $x = 0$ in this equation we see that

$$u(t, 0) = 0,$$

so the Dirichlet boundary condition is satisfied. So the restriction of this u to $\mathbf{R} \times [0, +\infty)$ is a solution to the original, unextended, initial value problem.

The preceding argument didn't give an explicit solution, since none was required, but it's very easy to obtain one if we want it. D'Alembert gives

$$u(t, x) = \frac{1}{2} f(x - ct) + \frac{1}{2} f(x + ct) + \frac{1}{2c} \int_{x-ct}^{x+ct} g(y)\, dy.$$

Here f, g and u are the extended functions but if $x \geq c|t|$ all the points where we evaluate them are in the original region so the formula above also works as an explicit formula for the original problem. If $ct > x \geq 0$ then we write

$$f(x - ct) = -f(ct - x)$$

and split the integral

$$\int_{x-ct}^{x+ct} g(y)\, dy = \int_{x-ct}^{0} g(y)\, dy + \int_{0}^{x+ct} g(y)\, dy.$$

In the first integral we write

$$g(y) = -g(-y)$$

to get

$$\int_{x-ct}^{0} g(y)\, dy = -\int_{x-ct}^{0} g(-y)\, dy$$

or, after a change of variable,

$$\int_{x-ct}^{0} g(y)\, dy = -\int_{0}^{ct-x} g(y)\, dy.$$

So

$$\int_{x-ct}^{x+ct} g(y)\, dy = -\int_{0}^{ct-x} g(y)\, dy + \int_{0}^{x+ct} g(y)\, dy = \int_{ct-x}^{x+ct} g(y)\, dy.$$

The explicit solution formula in this case is therefore

$$u(t, x) = -\frac{1}{2} f(ct - x) + \frac{1}{2} f(x + ct) + \frac{1}{2c} \int_{ct-x}^{x+ct} g(y)\, dy.$$

Here again we are evaluating all functions in the original domain so these are the unextended functions and so this makes sense as a solution to the original problem. A similar argument in the case $ct < -x$ gives

$$u(t,x) = \frac{1}{2}f(x-ct) - \frac{1}{2}f(-x-ct) - \frac{1}{2c}\int_{-x-ct}^{x-ct} g(y)\,dy.$$

(c) For f and g satisfying the conditions you gave in the first part prove that there is at most one classical solution.

Solution: There are at least three ways to do this. You can use Green's theorem in an appropriately shaped regions to show that any solution must be given by the formulae formulae in the solution to the previous part. Or you can extend to two hypothetical solutions to $\mathbf{R}^2$ by making them odd functions x, show that the extensions satisfy the initial value problem for the wave equation in $\mathbf{R}^2$ with the same initial data, and then refer to the uniqueness theorem for that problem. Or you can use the usual energy conservation and linearity argument: If there were two solutions their difference would be a solution with zero initial data, hence zero initial energy, and so would have zero energy for all time, which is possible only if it's the zero solution.

Which of these options should we choose? The first option is straightforward but it's messy since we need to choose three different regions for the three different cases. The second option is not as easy as it might seem. The difficulty is in showing that the second partial derivative $\partial^2 u/\partial x^2$ for the extended u exists at the boundary $x = 0$. Once you realise this isn't obvious it's not too difficult to prove, but it can be avoided by choosing the third option. Energy conservation for the Dirichlet problem on $\mathbf{R} \times [a, +\infty)$ was already proved in the notes so we just use that with $a = 0$ and we are done.

2. Show that

$$(Ju)(t,x) = \frac{1}{t^{1/2}} \exp\left(-\frac{x^2}{4kt}\right) u(-1/t, x/t)$$

is a symmetry of the diffusion equation.

Solution: Differentiating repeatedly,

$$\begin{aligned}\frac{\partial Ju}{\partial t}(t,x) &= \frac{\exp\left(-\frac{x^2}{4kt}\right)}{t^{5/2}} \frac{\partial u}{\partial t}(-1/t, x/t) - \frac{x\exp\left(-\frac{x^2}{4kt}\right)}{t^{5/2}} \frac{\partial u}{\partial x}(-1/t, x/t) \\ &\quad - \frac{\exp\left(-\frac{x^2}{4kt}\right)}{2t^{3/2}} u(-1/t, x/t) + \frac{x^2\exp\left(-\frac{x^2}{4kt}\right)}{4kt^{5/2}} u(-1/t, x/t),\end{aligned}$$

$$\frac{\partial Ju}{\partial x}(t,x) = \frac{\exp\left(-\frac{x^2}{4kt}\right)}{t^{3/2}} \frac{\partial u}{\partial x}(-1/t, x/t) - \frac{x\exp\left(-\frac{x^2}{4kt}\right)}{2kt^{3/2}} u(-1/t, x/t),$$

and

$$\begin{aligned}\frac{\partial^2 Ju}{\partial x^2}(t,x) &= \frac{\exp\left(-\frac{x^2}{4kt}\right)}{t^{5/2}} \frac{\partial^2 u}{\partial x^2}(-1/t, x/t) - \frac{x\exp\left(-\frac{x^2}{4kt}\right)}{kt^{5/2}} \frac{\partial u}{\partial x}(-1/t, x/t) \\ &\quad - \frac{\exp\left(-\frac{x^2}{4kt}\right)}{2kt^{3/2}} u(-1/t, x/t) + \frac{x^2\exp\left(-\frac{x^2}{4kt}\right)}{4k^2t^{5/2}} u(-1/t, x/t),\end{aligned}$$

so

$$\begin{aligned}&\frac{\partial Ju}{\partial t}(t,x) - k\frac{\partial^2 Ju}{\partial x^2}(t,x) \\ &\quad = \frac{\exp\left(-\frac{x^2}{4kt}\right)}{t^{5/2}} \left(\frac{\partial u}{\partial t}(-1/t, x/t) - k\frac{\partial^2 u}{\partial x^2}(-1/t, x/t)\right).\end{aligned}$$

It follows that Ju satisfies the diffusion equation everywhere it's defined if and only u satisfies the diffusion equation everywhere it's defined.

3. (a) In the notes we saw that

$$(S_{\alpha,\lambda}u)(t,x) = \lambda u(t/\alpha^2, x/\alpha)$$

is a symmetry for all non-zero α and λ. In particular, for any p and any positive α

$$(S_{\alpha,\alpha^p}u)(t,x) = \alpha^p u(t/\alpha^2, x/\alpha)$$

is a symmetry. Show that for each p the following two conditions area equivalent:

- u is invariant under the transformations S_{α,α^p} for all positive α,
- There is a function φ such that

$$u(t,x) = t^{p/2}\varphi(x/\sqrt{kt})$$

Note that we're not assuming, for the moment, that u is a solution of the diffusion equation.

Solution: Suppose that u is invariant under the transformations S_{α,α^p} for all positive α, i.e. that $S_{\alpha,\alpha^p}u = u$. Then

$$u(t,x) = \alpha^p u(t/\alpha^2, x/\alpha)$$

for all α and so, in particular, for $\alpha = \sqrt{kt}$:

$$u(t,x) = (kt)^{p/2} u(1/k, x/\sqrt{kt})$$

Thus

$$u(t,x) = t^{p/2}\varphi(x/\sqrt{kt})$$

with

$$\varphi(y) = k^{p/2} u(1/k, y).$$

This is the same argument that appeared in the notes in the special case $p = 0$.

Conversely, suppose

$$u(t,x) = t^{p/2}\varphi(x/\sqrt{kt})$$

Then

$$(S_{\alpha,\alpha^p}u)(t,x) = \alpha^p u(t/\alpha^2, x/\alpha) = \alpha^p (t/\alpha^2)^{p/2}\varphi\left(\frac{x/\alpha}{\sqrt{kt/\alpha^2}}\right)$$
$$= t^{p/2}\varphi(x/\sqrt{kt}) = u(t,x),$$

so $S_{\alpha,\alpha^p}u = u$. In other words u is invariant under the transformations S_{α,α^p} for all positive α.

(b) What ordinary differential equation does the function φ in the preceding part need to satisfy in order for u to be a solution of the diffusion equation.

Solution: If
$$u(t,x) = t^{p/2}\varphi(x/\sqrt{kt})$$
then, by the chain rule,

$$\frac{\partial u}{\partial t}(t,x) = -\frac{1}{2}k^{-1/2}t^{(p-3)/2}x\varphi'(x/\sqrt{kt}) + \frac{p}{2}t^{(p-2)/2}\varphi(x/\sqrt{kt}),$$

$$\frac{\partial u}{\partial x}(t,x) = k^{-1/2}t^{(p-1)/2}\varphi'(x/\sqrt{kt}),$$

and
$$\frac{\partial^2 u}{\partial x^2}(t,x) = k^{-1}t^{(p-2)/2}\varphi''(x/\sqrt{kt}).$$

Then

$$\frac{\partial u}{\partial t}(t,x) - k\frac{\partial^2 u}{\partial x^2}(t,x)$$
$$= -t^{(p-2)/2}\left(\varphi''(x/\sqrt{kt}) + \frac{1}{2}\frac{x}{\sqrt{kt}}\varphi'(x/\sqrt{kt}) - \frac{p}{2}\varphi(x/\sqrt{kt})\right)$$

So u satisfies the diffusion equation is satisfied if and only if φ satisfies the equation

$$\varphi''(y) + \frac{y}{2}\varphi'(y) - \frac{p}{2}\varphi(y) = 0.$$

(c) Show that when p is a non-negative integer the equation has a solution which is a polynomial of degree p.

Solution: One option is to substitute

$$\varphi(y) = \sum_{j=0}^{p} c_j y^j$$

into the differential equation and see what conditions the coefficients need to satisfy.

$$\varphi'(y) = \sum_{j=1}^{p} j c_j y^{j-1}$$

and

$$\varphi''(y) = \sum_{j=2}^{p} j(j-1) c_j y^{j-2}$$

so

$$z\varphi'(y) = \sum_{j=1}^{p} j c_j y^j = \sum_{j=0}^{p} j c_j y^j.$$

The two sums are the same, since the $j = 0$ summand is zero. We can change indices in the sum for the second derivative to get

$$\varphi''(y) = \sum_{j=0}^{p-2} (j+1)(j+2) c_{j+2} y^j.$$

If we set $c_k = 0$ for $k > j$, which makes sense since those coefficients are all zero, then we can equally well write this as

$$\varphi''(y) = \sum_{j=0}^{p} (j+1)(j+2) c_{j+2} y^j.$$

So

$$\begin{aligned} &\varphi''(y) + \frac{y}{2}\varphi'(y) - \frac{p}{2}\varphi(y) \\ &\quad = \sum_{j=0}^{p} \left[(j+1)(j+2) c_{j+2} + \frac{1}{2}(j-p) c_j \right] z^j = 0. \end{aligned}$$

A polynomial is zero if and only if all of its coefficients are so

$$\varphi''(y) + \frac{y}{2}\varphi'(y) - \frac{p}{2}\varphi(y) = 0$$

if and only if

$$2(j+1)(j+2)c_{j+2} = (p-j)c_j.$$

for all j. For $j \geq p$ this just says $0 = 0$ and so doesn't restrict the choice of coefficients in any way. For $j = p-1$ it says $0 = c_{p-1}$, so that coefficient must be zero. For $0 \leq j \leq p-2$ it determines uniquely c_j once c_{j+2} is known. So we can specify c_p arbitrarily and then all the other coefficients are determined.