MAU34215 Assignment 1
Due 1 October 2025
Solutions

1. What are the orders of the following differential equations? Which of them are linear?

   - Helmholtz equation:
   $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + u = 0,$$
   - Biharmonic equation:
   $$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0,$$
   - Eikonal equation:
   $$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - 1 = 0,$$
   - Euler-Tricomi equation:
   $$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial y^2} = 0,$$
   - Minimal surface equation:
   $$\left[1 + \left(\frac{\partial u}{\partial y}\right)^2\right] \frac{\partial^2 u}{\partial x^2} - 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x \partial y} + \left[1 + \left(\frac{\partial u}{\partial x}\right)^2\right] \frac{\partial^2 u}{\partial y^2} = 0,$$
   - Aller-Lytkov equation:
   $$a\frac{\partial^3 w}{\partial t \partial x^2} + d\frac{\partial^2 w}{\partial x^2} - \frac{\partial w}{\partial t} = 0.$$

   Solution: The eikonal equation is of first order. The Helmholtz equation, Euler-Tricomi equation and minimal surface equation are all of second order. The Aller-Lytkov equation is of third order. The biharmonic equation is of fourth order. All but the eikonal equation and minimal surface equation are linear.

2. (a) Show that if the initial data f and g are infinitely differentiable then so is the solution u to the initial value problem for the wave equation.

Solution: Integrating an infinitely differentiable function gives an infinitely differentiable function so the functions φ and ψ defined in the notes by

$$\varphi(z) = \frac{1}{2} f(z) + \frac{1}{2c} \int_z^p g(y)\, dy,$$
$$\psi(z) = \frac{1}{2} f(z) + \frac{1}{2c} \int_p^z g(y)\, dy$$

are infinitely differentiable. Then

$$u(t, x) = \varphi(x + cs - ct) + \psi(x - cs + ct)$$

is also infinitely differentiable.

(b) Show the following sort of converse: If u is infinitely differentiable for $t > s$ then f and g are infinitely differentiable.

Solution: Choose a $\tau > s$ and let

$$\tilde{f}(x) = u(\tau, x), \quad \tilde{g}(x) = \frac{\partial u}{\partial t}(\tau, x).$$

$\tilde{f}$ is the restriction of the function u from the set $(s, +\infty) \times \mathbf{R}$, where it is infinitely differentiable, to the subset $\{\tau\}$, and so is infinitely differentiable. Similarly, $\tilde{g}$ is the restriction of the infinitely differentiable function $\partial u / \partial t$, and so is infinitely differentiable. The initial value problem

$$\frac{\partial^2 \tilde{u}}{\partial t^2} - c^2 \frac{\partial^2 \tilde{u}}{\partial x^2} = 0,$$
$$\tilde{u}(\tau, x) = \tilde{f}(x), \quad \frac{\partial u}{\partial t}(\tau, x) = \tilde{g}(x)$$

has at most one solution by the uniqueness theorem from the notes. If it has a solution then that solution is infinitely differentiable for all t by the previous part of this problem. But we know it has a solution, namely $\tilde{u} = u$, so u infinitely differentiable not just for $t > s$, as was assumed, but for all values of t. $\partial u / \partial t$ is also then infinitely differentiable. But then f and g, as the restrictions of u and $\partial u / \partial t$ to $\{s\} \times \mathbf{R}$, are also infinitely differentiable.

3. Suppose that a solution of the wave equation on the interval $[a, b]$ satisfies the Robin boundary conditions

$$\frac{\partial u}{\partial x}(t,a) - \alpha u(t,a) = 0, \quad \frac{\partial u}{\partial x}(t,b) + \beta u(t,b) = 0.$$

   (a) Show that

$$\frac{\alpha c^2}{2} u(t,a)^2 + \int_a^b \left[\frac{1}{2} \left(\frac{\partial u}{\partial t}(t,x) \right)^2 + \frac{c^2}{2} \left(\frac{\partial u}{\partial x}(t,x) \right)^2 \right] dx + \frac{\beta c^2}{2} u(t,b)^2$$

   is independent of t.

   Solution: The argument follows the one given for the Dirichlet and Neumann conditions in the notes. We define functions

$$p = \frac{1}{2} \left(\frac{\partial u}{\partial t} \right)^2 - \frac{c^2}{2} \left(\frac{\partial u}{\partial t} \right)^2$$

   and

$$q = c^2 \frac{\partial u}{\partial t} \frac{\partial u}{\partial x}$$

   in the rectangle

$$R = [t_1, t_2] \times [a, b].$$

   Then

$$\int_a^b p(t_1, x)\, dx + \int_{t_1}^{t_2} q(t,b)\, dt - \int_a^b p(t_2, x)\, dx - \int_{t_1}^{t_2} q(t,a)\, dt$$
$$= \iint_R \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial t} \right) dA$$

   Now

$$\frac{\partial q}{\partial x} - \frac{\partial p}{\partial t} = -\left(\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} \right) \frac{\partial u}{\partial t} = 0$$

   because u is a solution of the wave equation. Also,

$$\int_a^b p(t_j, x)\, dx = \int_a^b \left[\frac{1}{2} \left(\frac{\partial u}{\partial t}(t_j, x) \right)^2 + \frac{c^2}{2} \left(\frac{\partial u}{\partial x}(t_j, x) \right)^2 \right] dx.$$

   So far everything has proceeded exactly as for the Dirichlet or Neumann boundary conditions. In those cases the next step was to say that the two q integrals vanish because of the boundary conditions. That's no longer true here though. Instead we have

$$\begin{aligned}
q(t,a) &= c^2 \frac{\partial u}{\partial t}(t,a)\frac{\partial u}{\partial x}(t,a) \\
&= \alpha c^2 u(t,a)\frac{\partial u}{\partial t}(t,a) \\
&= \frac{\alpha c^2}{2}\frac{d}{dt}u(t,a)^2.
\end{aligned}$$

The fundamental theorem of calculus therefore gives

$$-\int_{t_1}^{t_2} q(t,a)\,dt = \frac{\alpha c^2}{2}\left[u(t_1,a)^2 - u(t_2,a)^2\right].$$

Similarly,

$$\int_{t_1}^{t_2} q(t,b)\,dt = \frac{\beta c^2}{2}\left[u(t_1,b)^2 - u(t_2,b)^2\right].$$

Combining the equations above we see that

$$\frac{\alpha c^2}{2}u(t_1,a)^2 + \int_a^b \left[\frac{1}{2}\left(\frac{\partial u}{\partial t}(t_1,x)\right)^2 + \frac{c^2}{2}\left(\frac{\partial u}{\partial x}(t_1,x)\right)^2\right]dx + \frac{\beta c^2}{2}u(t_1,b)^2$$

is equal to

$$\frac{\alpha c^2}{2}u(t_2,a)^2 + \int_a^b \left[\frac{1}{2}\left(\frac{\partial u}{\partial t}(t_2,x)\right)^2 + \frac{c^2}{2}\left(\frac{\partial u}{\partial x}(t_2,x)\right)^2\right]dx + \frac{\beta c^2}{2}u(t_2,b)^2.$$

(b) Prove uniqueness of solutions to the initial value problem with Robin boundary conditions under the assumption that α and β are non-negative.

Solution: Suppose u_1 and u_2 are solutions to the initial value problem with Robin boundary conditions and the same initial data. Then $u = u_1 - u_2$ is a solution with zero initial data. It follows that

$$\frac{\alpha c^2}{2}u(t,a)^2 + \int_a^b \left[\frac{1}{2}\left(\frac{\partial u}{\partial t}(t,x)\right)^2 + \frac{c^2}{2}\left(\frac{\partial u}{\partial x}(t,x)\right)^2\right]dx + \frac{\beta c^2}{2}u(t,b)^2$$

is zero initially and hence is zero for all t. Since each summand is non-negative they must all be zero. The integrand is non-negative and continuous so if the integral is zero then the integrand is zero everywhere. But this means that u is zero everywhere.

(c) Does your argument for the preceding part work without that assumption?

Solution: No. Without that assumption we can't say that the first and last terms are non-negative. Of course this doesn't mean that uniqueness of solutions necessarily fails, merely that this argument can't be used to prove it.