MAU34215 Assignment 4
Due 26 November 2025

1. Prove that the following are symmetries of Burgers' equation.

   (a)
   $$\tilde{u}(t,x) = u(t, x - vt) + v$$
   for all v.

   (b)
   $$\tilde{u}(t,x) = u(-t,-x).$$

   (c)
   $$u(t,x) = \mu u(t/\mu, x/\mu^2)$$
   for all non-zero μ.

2. We proved existence and uniqueness for the Dirichlet problem for the unit disc, but not for the Neumann problem. The Neumann problem is, given a continuous function g on the unit circle, to find a continuously differentiable function u on closed unit disc which is twice continuously differentiable in the open unit disc and satisfies the Laplace equation there, and whose radial derivative,
   $$\frac{x}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial x} + \frac{y}{\sqrt{x^2+y^2}}\frac{\partial u}{\partial y},$$
   is equal to g on the unit circle.

   (a) Show that the Neumann problem does not have unique solutions by giving a function g and two distinct solutions, u_1 and u_2 to the Neumann problem for this g.

   Hint: There are a lot of possible choices but some of them are extremely simple, so just look for very simple solutions of the Laplace equation and see whether any work.

   (b) Show that there are no solutions to the Neumann problem unless
   $$\int_{-\pi}^{\pi} g(\cos\theta, \sin\theta)\, d\theta = 0.$$

Hint: Apply Green's theorem to the functions

$$p(x, y) = \frac{\partial u}{\partial y}, \quad q(x, y) = -\frac{\partial u}{\partial x}$$

in an appropriate region.

3. The region

$$R = \left\{(x, y) \in \mathbf{R}^2 \colon x \geq 0, y \geq 0, x^2 + y^2 \leq 1\right\}$$

in the plane has a boundary consisting of the three curves

$$C_1 = \left\{(x, y) \in \mathbf{R}^2 \colon x \geq 0, y = 0, x^2 + y^2 \leq 1\right\},$$

$$C_2 = \left\{(x, y) \in \mathbf{R}^2 \colon x \geq 0, y \geq 0, x^2 + y^2 = 1\right\}$$

and

$$C_3 = \left\{(x, y) \in \mathbf{R}^2 \colon x = 0, y \geq 0, x^2 + y^2 \leq 1\right\}.$$

There is a transformation of the plane which is a symmetry of the Laplace equation and maps R to itself in such a way that C_1 is mapped to C_2, C_2 is mapped to C_3 and C_3 is mapped to C_1. The goal of this problem is to find that transformation.

(a) As described in the notes, Lorentz matrices correspond to symmetries of Laplace. The symmetry we're looking for will map the x axis to the unit circle, the unit circle to the y axis and the y axis to the x axis. What conditions do we need on our Lorentz matrix to accomplish this.

(b) Find a Lorentz matrix with the required properties.

(c) What is the corresponding symmetry of the Laplace equation?