MAU34215 Assignment 3
Due 5 November 2025

1. In the notes and in lecture we mostly restricted our attention to bounded solutions and bounded initial data. There were good reasons for this but unbounded data and solutions do occur in applications. It's still possible to draw some useful conclusions provided the data aren't too badly unbounded.

   (a) Find solutions to the usual initial value problem

   $$\frac{\partial u}{\partial t}(t,x) - k\frac{\partial^2 u}{\partial x^2}(t,x) = 0$$

   for $t \geq 0$ and $x \in \mathbf{R}$ with

   $$u(0,x) = f(x)$$

   with the following initial data:

   i.
   $$f(x) = \exp(rx),$$

   ii.
   $$f(x) = \exp(-rx),$$

   iii.
   $$f(x) = \cosh(rx).$$

   Here r is a real number.

   Note: If we exclude the rather uninteresting case of $r = 0$ then these f's are unbounded, so few of the the theorems from lecture or the notes apply. Depending on how you solve the problem you may therefore need to check some things which you could otherwise use those theorems for.

   (b) Show that if

   $$|f(x)| \leq C\exp(rx)$$

   or

   $$|f(x)| \leq C\exp(-rx)$$

for some real C and r then the initial value problem has a unique solution u satisfying

$$|u(t,x)| \leq C \exp(kr^2 t + rx)$$

or

$$|u(t,x)| \leq C \exp(kr^2 t - rx),$$

respectively, for all $t \geq 0$ and $x \in \mathbf{R}$.

Hint: It's certainly possible to follow the existence and uniqueness proofs from the notes making various changes to fit these new hypotheses and conclusions but those proofs were long and complicated. Can you use symmetries instead?

(c) Show that if

$$|f(x)| \leq C \cosh(rx)$$

for some real C and r then the initial value problem has a solution u satisfying

$$|u(t,x)| \leq C \exp(kr^2 t) \cosh(rx).$$

for all $t \geq 0$ and $x \in \mathbf{R}$.

Hint: Uniqueness is still true with these hypotheses but I haven't asked you prove it, only existence. Again, it's possible, but unnecessary, to follow the existence proof from the notes. Can you use linearity instead?

2. (a) Suppose u is a solution to the initial value problem

$$\frac{\partial u}{\partial t}(t,x) - k\frac{\partial^2 u}{\partial x^2}(t,x) = 0, \quad u(0,x) = f(x)$$

where f is bounded and continuous. Prove that

$$\int_{-\infty}^{+\infty} |u(t,x)|\, dx \leq \int_{-\infty}^{+\infty} |f(x)|\, dx.$$

(b) With hypotheses as above prove that the inequality above is strict if and only if there are points x_1 and x_2 where $f(x_1) > 0$ and $f(x_2) < 0$.

3. Solve Burgers' equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0$$

with initial data

$$u(0, x) = x + \sqrt{1 + x^2}.$$