MAU34215 Assignment 2
Due 15 October 2025

1. Consider the initial value problem

$$u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = g(x)$$

for the wave equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

in $\mathbf{R} \times [0, +\infty)$ with Dirichlet boundary conditions at $x = 0$, i.e.

$$u(t, 0) = 0.$$

   (a) What conditions on f and g are needed to obtain a classical solution.

   (b) For f and g satisfying the conditions you gave in the first part prove that there is at least one classical solution.

   (c) For f and g satisfying the conditions you gave in the first part prove that there is at most one classical solution.

2. Show that

$$(Ju)(t,x) = \frac{1}{t^{1/2}} \exp\left(-\frac{x^2}{4kt}\right) u(-1/t, x/t)$$

is a symmetry of the diffusion equation.

3. (a) In the notes we saw that

$$(S_{\alpha,\lambda}u)(t,x) = \lambda u(t/\alpha^2, x/\alpha)$$

is a symmetry for all non-zero α and λ. In particular, for any p and any positive α

$$(S_{\alpha,\alpha^p}u)(t,x) = \alpha^p u(t/\alpha^2, x/\alpha)$$

is a symmetry. Show that for each p the following two conditions area equivalent:

- u is invariant under the transformations S_{α,α^p} for all positive α,
- There is a function φ such that

$$u(t,x) = t^{p/2}\varphi(x/\sqrt{kt})$$

Note that we're not assuming, for the moment, that u is a solution of the diffusion equation.

(b) What ordinary differential equation does the function φ in the preceding part need to satisfy in order for u to be a solution of the diffusion equation.

(c) Show that when p is a non-negative integer the equation has a solution which is a polynomial of degree p.