

MAU34215 Assignment 4
Due 2026-11-18

1. (a) The function

$$f(x) = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0 \end{cases}$$

is bounded but not continuous. Ignore this and solve the boundary value problem

$$u(x, 0) = f(x)$$

for the Laplace equation in the upper half plane as if it were continuous.

- (b) Because the Poisson formula was only derived for solutions continuous on the closed upper half-plane we can't just assume that the formal calculation above gives a bounded solution of the Laplace equation. Check that it does.
- (c) Consider the function defined by the Poisson formula for $y = 0$ and the boundary condition for $y = 0$. This function can't be continuous at $(0, 0)$ because if it were then its restriction to the x axis, which is f , would be continuous. Is it continuous at other points?
2. (a) Prove that if u is a continuous on the closed disc of radius r about the point (a, b) and satisfies the Laplace equation in the open disc then

$$\|\nabla u(a, b)\| \leq \frac{4}{\pi} \frac{\max |u|}{r},$$

where the maximum is over the closed disc.

It's probably easiest to do this first in the special case $(a, b) = 0$ and $r = 1$. Don't worry too much about getting the precise constant in front as it won't affect later parts of the problem.

- (b) Suppose that u is a bounded solution to the Laplace equation on all of $\mathbf{R}^2$. Prove that u is constant. You may use the result from the previous part even if you didn't manage to prove it.

3. Suppose φ is a differentiable convex function and u is a bounded function on the closed upper half-plane satisfying the Laplace equation in the open upper half-plane and the boundary condition $u(x, 0) = f(x)$ for all x . Prove that

$$\int_{-\infty}^{+\infty} \varphi(u(x, y)) \, dx \leq \int_{-\infty}^{+\infty} \varphi(f(x)) \, dx$$

for all $y > 0$. The method is the same one we used for the corresponding theorem about the diffusion equation.