MAU34215 Assignment 2
Due 2026-10-14

1. Consider the initial value problem on the half line $[0, +\infty)$ with the Dirichlet condition $u(t, 0) = 0$ on the boundary and the specific initial conditions

$$u(0, x) = \sin^2 x, \quad \frac{\partial u}{\partial t}(0, x) = 0$$

for $x \in [0, \infty)$. Find the unique solution satisfying these conditions for $t \geq 0$.

2. Consider the Klein-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + u = 0$$

on the region $\mathbf{R} \times [a, b]$ with Dirichlet boundary conditions at both ends

$$u(t, a) = 0 = u(t, b).$$

Define the energy by

$$e(t) = \int_a^b \left[\frac{1}{2} \left(\frac{\partial u}{\partial t} \right)^2 + \frac{c^2}{2} \left(\frac{\partial u}{\partial x} \right)^2 + \frac{m^2}{2} u^2 \right] dx$$

(a) Show that energy is conserved, i.e. that $e(t_1) = e(t_2)$ for all $t_1, t_2 \in \mathbf{R}$.

(b) Show that the initial value problem

$$u(s, x) = f(s), \quad \frac{\partial u}{\partial x}(x, s) = g(s)$$

for the Klein-Gordon equation has at most one solution. You do not need to prove existence, i.e. that there is at least one solution. Also, you may use the result of the preceding part even if you didn't succeed in proving it.

(c) Suppose the sign of the zeroeth order term is reversed, i.e. that the equation is

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} - m^2 u = 0$$

and the energy is defined by

$$e(t) = \int_a^b \left[\frac{1}{2} \left(\frac{\partial u}{\partial t} \right)^2 + \frac{c^2}{2} \left(\frac{\partial u}{\partial x} \right)^2 - \frac{m^2}{2} u^2 \right] dx$$

Does your proof of energy conservation go through with appropriate changes? If so, what changes are needed? If not, why not? Does your proof of uniqueness go through with appropriate changes? If so, what changes are needed? If not, why not?

3. In the notes we saw that

$$(S_{\alpha,\lambda} u)(t, x) = \lambda u(t/\alpha^2, x/\alpha)$$

is a symmetry for all non-zero α and λ. In particular, for any p and any positive α

$$(S_{\alpha,\alpha^p} u)(t, x) = \alpha^p u(t/\alpha^2, x/\alpha)$$

is a symmetry.

(a) Show that for each p the following two conditions area equivalent:

- u is invariant under the transformations S_{α,α^p} for all positive α,
- There is a function φ such that

$$u(t, x) = t^{p/2} \varphi(x/\sqrt{kt})$$

Note that we're not assuming, for the moment, that u is a solution of the diffusion equation.

(b) What ordinary differential equation does the function φ in the preceding part need to satisfy in order for u to be a solution of the diffusion equation.

(c) Show that when p is a non-negative integer the equation has a solution which is a polynomial of degree p.