

MAU34215 Assignment 1
Due 2026-09-30

1. For the following equations list the dependent and independent variables, give their order, whether they are linear or non-linear, and whether they are ordinary or partial differential equations.

In some cases the answer may be different for special values of the parameters. In those cases you can just give the answer for the generic case.

- (a) Helmholtz:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + k^2 u = 0$$

Solution: Dependent variable: u . Independent variables: x , y and z . This is a second order linear partial differential equation.

- (b) Eikonal:

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 = 1$$

Solution: Dependent variable: u . Independent variables: x , y and z . This is a first order non-linear partial differential equation.

- (c) Picard-Fuchs:

$$\frac{d^2 f}{dj^2} + \frac{1 - 1968j + 2654208j^2}{4j^2(1 - 1728j)^2} f = 0$$

Solution: Dependent variable: f . Independent variable: j . This is a second order linear ordinary differential equation.

- (d) Painlevé II:

$$\frac{d^2 y}{dt^2} = 2y^3 + ty + \alpha$$

Solution: Dependent variable: y . Independent variable: t . This is a second order non-linear ordinary differential equation.

- (e) Lane-Emden:

$$\xi \frac{d^2 \theta}{d\xi^2} + 2 \frac{d\theta}{d\xi} + \xi \theta^n = 0$$

Solution: Dependent variable: θ . Independent variable: ξ . This is a second order non-linear ordinary differential equation.

(f) Tzitzeica:

$$w \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} - w^3 + 1 = 0$$

Solution: Dependent variable: w . Independent variables: x and y . This is a second order non-linear partial differential equation.

(g) Camassa-Holm:

$$\frac{\partial u}{\partial t} + 2\kappa \frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial t \partial x^2} + 3u \frac{\partial u}{\partial x} - 2 \frac{\partial u}{\partial x} \frac{\partial^2 u}{\partial x^2} + u \frac{\partial^3 u}{\partial x^3} = 0$$

Solution: Dependent variable: u . Independent variables: t and x . This is a third order non-linear partial differential equation.

2. Which of the symmetries we found for the wave equation are also symmetries of the Klein-Gordon equation?

Solution: Since the Klein-Gordon equation is linear it has a scaling symmetry in the dependent variable and also addition of any particular solution. For the other symmetries of the wave equation we could check each of them individually but there's an easier way. All of them are of the form

$$\tilde{u}(x, y) = u(p(x, y), q(x, y))$$

for some functions p and q and so, as shown in the notes, satisfy either the pair of equations

$$\frac{\partial p}{\partial t} = \frac{\partial q}{\partial x}, \quad \frac{\partial q}{\partial t} = c^2 \frac{\partial p}{\partial x}$$

or the pair

$$\frac{\partial p}{\partial t} = \frac{\partial q}{\partial x}, \quad \frac{\partial q}{\partial t} = c^2 \frac{\partial p}{\partial x}$$

and have

$$\left(\frac{\partial^2 \tilde{u}}{\partial t^2} - c^2 \frac{\partial^2 \tilde{u}}{\partial x^2} \right) (x, y) = \pm \left(\frac{\partial p}{\partial t} \frac{\partial q}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial q}{\partial t} \right) (x, y) \left(\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} \right) (p(x, y), q(x, y))$$

where the sign is positive for the first set of equations and negative for the second. We therefore have

$$\begin{aligned} \left(\frac{\partial^2 \tilde{u}}{\partial t^2} - c^2 \frac{\partial^2 \tilde{u}}{\partial x^2} + m^2 u \right) (x, y) = \pm \left(\frac{\partial p}{\partial t} \frac{\partial q}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial q}{\partial t} \right) (x, y) \\ \left(\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + m^2 u \right) (p(x, y), q(x, y)) \\ + m^2 u(p(x, y), q(x, y)) \end{aligned}$$

This will be a symmetry of the Klein-Gordon equation if and only if the right hand side is a multiple of

$$\left(\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} + m^2 u \right) (p(x, y), q(x, y)),$$

which happens if and only if

$$\left(\frac{\partial p}{\partial t} \frac{\partial q}{\partial x} - \frac{\partial p}{\partial x} \frac{\partial q}{\partial t} \right) (x, y) = \pm 1$$

at all points (x, y) in the domain of u . The sign here is as above. It therefore suffices to identify the functions p and q , compute their first derivative, check which pair of equations is satisfied, compute the jacobian determinant, and check whether it's equal to ± 1 with the choice of sign corresponding to that pair. This is a lot of steps, but each of them individually is easy.

We have the following table for all of the symmetries except inversion.

	Jacobian	pair	determinant	K-G symmetry?
S_α	$\begin{bmatrix} 1/\alpha & 0 \\ 0 & 1/\alpha \end{bmatrix}$	first	$1/\alpha^2$	only if $\alpha = \pm 1$
R	$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$	second	-1	yes
$T_{\tau, \xi}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	first	1	yes
L_v	$\frac{1}{\sqrt{1-v^2/c^2}} \begin{bmatrix} 1 & v/c^2 \\ v & 1 \end{bmatrix}$	first	1	yes

For the inversion symmetry J we have the Jacobian

$$\frac{1}{(c^2t^2 - x^2)^2} \begin{bmatrix} c^2t^2 + x^2 & -2tx \\ -c^22tx & c^2t^2 + x^2 \end{bmatrix}$$

which satisfies the first pair of equations but has determinant $\frac{1}{(c^2t^2 - x^2)^2}$, which is not equal to 1.

3. The following relate to the Lorentz transformations L_v from the notes and solutions in the region $ct > |x|$ in the plane.

- (a) Show that a function u is invariant under the Lorentz transformations if and only if it is of the form

$$u(t, x) = \varphi(c^2t^2 - x^2).$$

for some function φ on the positive real numbers.

Solution:

Invariance means $L_v u = u$ for all $v \in (-c, c)$, i.e. that

$$u(t, x) = u\left(\frac{t + vx/c^2}{\sqrt{1 - v^2/c^2}}, \frac{vt + x}{\sqrt{1 - v^2/c^2}}\right)$$

for all v , t and x . In particular this equation must hold for $v = -x/t$, which is the interval $(-c, c)$ because $ct > |x|$, so

$$u(t, x) = u\left(\frac{t - x^2/(c^2t)}{\sqrt{1 - x^2/(c^2t^2)}}, 0\right) = u\left(\frac{\sqrt{c^2t^2 - x^2}}{c}, 0\right)$$

so the equation

$$u(t, x) = \varphi(c^2t^2 - x^2).$$

holds with

$$\varphi(z) = u(\sqrt{z}/c, 0).$$

This is only half of what we need though. We also need to show that if

$$u(t, x) = \varphi(c^2t^2 - x^2).$$

then

$$u(t, x) = u\left(\frac{t + vx/c^2}{\sqrt{1 - v^2/c^2}}, \frac{vt + x}{\sqrt{1 - v^2/c^2}}\right)$$

for all v , t and x . The left hand side is obviously $\varphi(c^2t^2 - x^2)$.
The right hand side is

$$\varphi \left(c^2 \left(\frac{t + vx/c^2}{\sqrt{1 - v^2/c^2}} \right)^2 - \left(\frac{vt + x}{\sqrt{1 - v^2/c^2}} \right)^2 \right),$$

which is

$$\varphi \left(\frac{c^2(t + vx/c^2)^2 - (vt + x)^2}{1 - v^2/c^2} \right)$$

or $\varphi(c^2t^2 - x^2)$, as required.

- (b) Show that a continuously differentiable function u is invariant under the Lorentz transformations if and only if

$$x \frac{\partial u}{\partial t} + c^2 t \frac{\partial u}{\partial x} = 0$$

everywhere.

Solution: Since the equation

$$u(t, x) = u \left(\frac{t + vx/c^2}{\sqrt{1 - v^2/c^2}}, \frac{vt + x}{\sqrt{1 - v^2/c^2}} \right)$$

holds for all v in the interval $(-c, c)$ we can differentiate with respect v . This gives

$$0 = \frac{c}{(c^2 - v^2)^{3/2}} \left((x + vt) \frac{\partial u}{\partial t} + (vx + c^2 t) \frac{\partial u}{\partial x} \right) \left(\frac{t + vx/c^2}{\sqrt{1 - v^2/c^2}}, \frac{vt + x}{\sqrt{1 - v^2/c^2}} \right).$$

This holds for all v in the interval, but in particular it holds for $v = 0$, so

$$\frac{c}{(c^2 - v^2)^{3/2}} \left(x \frac{\partial u}{\partial t} + c^2 t \frac{\partial u}{\partial x} \right) (t, x) = 0.$$

The factor $c/(c^2 - v^2)^{3/2}$ is non-zero, so this equation holds if and only if the equation without that factor holds.

This is half of what we need. We also need to show that if

$$x \frac{\partial u}{\partial t} + c^2 t \frac{\partial u}{\partial x} = 0$$

holds everywhere then u is invariant. To show this we note that if the equation above holds everywhere then

$$\frac{c}{(c^2 - v^2)^{3/2}} \left((x + vt) \frac{\partial u}{\partial t} + (vx + c^2 t) \frac{\partial u}{\partial x} \right) \left(\frac{t + vx/c^2}{\sqrt{1 - v^2/c^2}}, \frac{vt + x}{\sqrt{1 - v^2/c^2}} \right) = 0.$$

The left hand side is, as we've already seen, the derivative of

$$u \left(\frac{t + vx/c^2}{\sqrt{1 - v^2/c^2}}, \frac{vt + x}{\sqrt{1 - v^2/c^2}} \right)$$

with respect to v , so if this is zero then this quantity is independent of v . It must therefore be equal to its value at $v = 0$, which is just $u(t, x)$. But this is exactly what we needed to prove.

(c) Find all Lorentz invariant solutions.

Solution: There are two ways to do this, using either of the previous parts. To use the first part we just substitute $\varphi(c^2 t^2 - x^2)$ in for $u(t, x)$ in the wave equation. We get

$$\frac{\partial u}{\partial t}(t, x) = 2c^2 t \varphi'(c^2 t^2 - x^2),$$

$$\frac{\partial u}{\partial x}(t, x) = 2x \varphi'(c^2 t^2 - x^2),$$

$$\frac{\partial^2 u}{\partial t^2}(t, x) = 4c^4 t^2 \varphi''(c^2 t^2 - x^2) - 2c^2 \varphi'(c^2 t^2 - x^2)$$

and

$$\frac{\partial^2 u}{\partial x^2}(t, x) = 4x^2 \varphi''(c^2 t^2 - x^2) - 2\varphi'(c^2 t^2 - x^2),$$

so

$$\frac{\partial^2 u}{\partial t^2}(t, x) - c^2 \frac{\partial^2 u}{\partial x^2}(t, x) = 4c^2 [(c^2 t^2 - v^2) \varphi''(c^2 t^2 - x^2) + \varphi'(c^2 t^2 - x^2)].$$

Therefore u satisfies the wave equation if and only if φ satisfies the ordinary differential equation

$$z\varphi''(z) + \varphi'(z) = 0.$$

The general solution of this equation is $\varphi(z) = \alpha \log z + \beta$ so the Lorentz invariant solutions of the wave equation are of the form

$$u(t, x) = \alpha \log(c^2 t^2 - x^2) + \beta.$$

An alternative solution method is to observe that

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} &= \frac{c^2}{c^2 t^2 - x^2} \left(t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \right)^2 u \\ &\quad - \frac{1}{c^2 t^2 - x^2} \left(x \frac{\partial}{\partial t} + c^2 t \frac{\partial}{\partial x} \right)^2 u. \end{aligned}$$

If u is Lorentz invariant then the second term on the right is zero by the previous part, so a Lorentz invariant u satisfies the wave equation if and only if it satisfies the simpler equation

$$\left(t \frac{\partial}{\partial t} + x \frac{\partial}{\partial x} \right)^2 u = 0.$$

This looks like a partial differential equation but it behaves more like an ordinary differential equation. The derivatives are directional derivatives on rays from the origin. In fact, if we remember the first part of the problem we only need to solve the equation on the time axis because we can recover the solution everywhere else from there. On the time axis this really is an ordinary differential equation, and is in fact essentially the same ordinary differential equation we met in the other solution method.