MA 3421 Assignment 8, Due 22 November 2018

Solutions

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1. The norm version of the Hahn-Banach theorem was proved in lecture and in the notes using Zorn's Lemma. Give a proof without using Zorn's Lemma in the special case where the Banach space is a Hilbert space.
   Solution: Suppose F is a subspace of a Hilbert space E and $f \in F'$. We can extend f to a linear function h on $H = \overline{F}$ by continuity,

   $$h(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n)$$

   for any sequence $(x_1, x_2, \ldots)$ in F. This is well defined since if

   $$\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$$

   then

   $$\begin{aligned}\lim_{n\to\infty} f(x_n) - \lim_{n\to\infty} f(y_n) &= \lim_{n\to\infty} [f(x_n) - f(y_n)] \\ &= \lim_{n\to\infty} [f(x_n - y_n)] \\ &= f\left(\lim_{n\to\infty} (x_n - y_n)\right) \\ &= f(0) = 0\end{aligned}$$

   so the value of h at a point in H is independent of the choice of approximating sequence. Also

   $$\begin{aligned}\left\| h\left(\lim_{n\to\infty} x_n\right)\right\| &= \left\|\lim_{n\to\infty} f(x_n)\right\| \\ &= \lim_{n\to\infty} \|f(x_n)\| \\ &\le \lim_{n\to\infty} \|f\|\|x_n\| \\ &= \|f\| \lim_{n\to\infty} \|x_n\|. \\ &= \|f\| \left\|\lim_{n\to\infty} x_n\right\|.\end{aligned}$$

   Every $z \in H$ is of the form

   $$z = \lim_{n\to\infty} x_n,$$

so

$$\|h\| \leq \|f\|.$$

But

$$f(x) = h(x)$$

for $x \in F$ and hence

$$|f(x)| = |h(x)| \leq \|h\|\|x\|.$$

This holds for all $x \in F$, so

$$\|f\| \leq \|h\|$$

and hence

$$\|h\| = \|f\|.$$

By the Riesz Representation Theorem there is a $w \in H$ such that

$$h(z) = (z|w)$$

for all $z \in H$ and

$$\|h\| = \|w\|.$$

We can then define

$$g(z) = (z|w)$$

for $z \in E$. Then

$$g(x) = h(x) = f(x)$$

when $x \in F$ and

$$\|g\| = \|w\| = \|h\| = \|f\|.$$

2. Show that $\mathbf{R} - \mathbf{Q}$ is not a countable union of closed sets.
Hint: Use the Baire Category Theorem.
Solution: Suppose

$$\mathbf{R} - \mathbf{Q} = \cup_{n=1}^{\infty} C_n$$

where C_n is closed. $\mathbf{Q}$ is countable, so let $\{q_1, q_2, \ldots\}$ be an enumeration of $\mathbf{Q}$. Define

$$S_n = \begin{cases} C_k & \text{if } n = 2k-1, \\ \{q_k\} & \text{if } n > 2k. \end{cases}$$

Then

$$\mathbf{R} = \cup_{n=1}^{\infty} S_n.$$

The Banach space $\mathbf{R}$ is a countable union of closed sets, so at least one contains an open ball, i.e. interval. It can't be a C_k, because then $\mathbf{R} - \mathbf{Q}$ would contain an open interval and it certainly can't be one of the $\{q_k\}$, so we have a contradiction.

3. Suppose that $A\colon H \to H$ is linear, where H is a Hilbert space, and that

$$(Ax|y) = (x|Ay)$$

for all A. Show that A is symmetric.
Hint: The problem is to show boundedness, since all the other properties of symmetric operators are true by assumption. Use the Closed Graph Theorem.
Solution: What we need to show is that if

$$\lim x_n = w, \quad \lim y_n = z$$

and

$$y_n = Ax_n$$

then

$$Aw = z.$$

For any $u \in H$,

$$(y_n|u) \to (z|u)$$

and

$$(y_n|u) = (Ax_n|u) = (x_n|Au) \to (w|Au) = (Aw|u)\,.$$

so

$$(z|u) = (Aw|u)\,.$$

This holds for all $u \in H$, so

$$z = Aw.$$

4. Suppose that $T_{j,k} \in \mathcal{L}(E, F)$ for each $j, k \in \mathbf{Z}^+$, where E and F are Banach spaces. It's trivially true that if there is an $x \in E$ such that for each $j \in \mathbf{Z}^+$, $T_{j,k}x$ is unbounded as a function of k then for each $j \in \mathbf{Z}^+$ there is an $x \in E$ such that $T_{j,k}$ is unbounded as a function of k. The converse is called the Principle of Condensation of Singularities: If for each $j \in \mathbf{Z}^+$ there is an $x \in E$ such that $T_{j,k}$ is unbounded as a function of k then there is an $x \in E$ such that for each $j \in \mathbf{Z}^+$, $T_{j,k}x$ is unbounded as a function of k. Prove it.
Hint: Consider

$$C_{j,k,l} = \{x \in E\colon \|T_{j,k}x\| \le l\}.$$

Is

$$\cup_{j \in \mathbf{Z}^+} \cup_{l \in \mathbf{Z}^+} \cap_{k \in \mathbf{Z}^+} C_{j,k,l} = E?$$

Use the Baire Category Theorem. You'll face the same problem that we faced in the proof of the Uniform Boundedness Theorem, that the open ball need not be centred at the origin, but you can solve it in the same way we did there.
Solution: Suppose on the contrary it's not true that there is an $x \in E$ such that for each $j \in \mathbf{Z}^+$, $T_{j,k}x$ is unbounded as a function of k. Then

for all $x \in E$ there is a $j \in \mathbf{Z}^+$ such that $T_{j,k}$ is bounded as a function of k. If there is a bound then there's one in $\mathbf{Z}^+$. In other words for all $x \in E$ there is a $j \in \mathbf{Z}^+$ and an $l \in \mathbf{Z}^+$ such that for all $k \in \mathbf{Z}^+$ we have $\|T_{j,k}x\| \leq l$. Yet another way to say this is that if $x \in E$ then

$$x \in \cup_{j \in \mathbf{Z}^+} \cup_{l \in \mathbf{Z}^+} \cap_{k \in \mathbf{Z}^+} C_{j,k,l}.$$

The converse is certainly true and sets are equal if and only if they have the same elements, so

$$\cup_{j \in \mathbf{Z}^+} \cup_{l \in \mathbf{Z}^+} \cap_{k \in \mathbf{Z}^+} C_{j,k,l} = E.$$

Now $C_{j,k,l}$ is closed and therefore so is

$$S_{j,l} = \cap_{k \in \mathbf{Z}^+} C_{j,k,l}.$$

Their union is E, so by the Baire Category theorem there is a $j \in \mathbf{Z}^+$ and an $l \in \mathbf{Z}^+$ such that $S_{j,l}$ contains an open ball, whose radius we'll call ρ and whose centre we'll call w. For any $x \in E - \{0\}$ set

$$y = w + \frac{\rho}{2\|x\|} x$$

and

$$z = w - \frac{\rho}{2\|x\|} x.$$

Then

$$\|y - w\| = \|z - w\| = \frac{\rho}{2} < \rho$$

and so

$$y, z \in S_{j,l}.$$

But then

$$\|T_{j,k}y\| \leq l, \quad \|T_{j,k}z\| \leq l$$

for all $k \in \mathbf{Z}^+$. Now

$$x = \frac{\|x\|}{\rho}(y - z),$$

so

$$\|T_{j,k}x\| \leq 2\frac{l}{\rho}\|x\|$$

for all $k \in \mathbf{Z}^+$. In other words, there is a $j \in \mathbf{Z}^+$ such that for all $x \in E$ $T_{j,k}x$ is bounded as a function of k, contrary to our hypothesis. Other than the hypotheses the only thing we assumed was that it is not true that there is an $x \in E$ such that for each $j \in \mathbf{Z}^+$, $T_{j,k}x$ is unbounded as a function of k. The contradiction shows that it is true.

5. We say that a sequence $(A_1, A_2, \ldots)$ in $\mathcal{L}(E, F)$ *converges strongly* to $C \in \mathcal{L}(E, F)$ if for all $x \in$ the sequence $(A_1x, A_2x, \ldots)$ in F converges to Cx.

(a) Show that if $(A_1, A_2, \ldots)$ converges to C then it converges strongly.

(b) Given an example of E, F, $(A_1, A_2, \ldots)$ and C for which $(A_1, A_2, \ldots)$ converges strongly to C but does not converge to C.[1]

Hint: The first part is easy. For the second part you might find it helpful to revise the discussion of weak convergence.
Solution:

(a) Let $x \in E$. Choose $\gamma > \|x\| \geq 0$ For any $\epsilon > 0$ we have $\epsilon/\gamma > 0$ so there is an N such that if $j > N$ then

$$\|A_j - C\| < \frac{\epsilon}{\gamma}.$$

But then if $j > N$ then

$$\|A_j x - Cx\| \leq \|A_j - C\| \|x\| < \epsilon$$

so $(A_1 x, A_2 x, \ldots)$ converges to Cx.

(b) Let $E = l^2$, $F = \mathbf{K}$,

$$A_n(x) = (x|e_n),$$

where e_n is the n'th standard basis vector, and $C = 0$. For any $x = (\xi_1, \xi_2, \ldots) \in l^2$ we have

$$A_n(x) = \xi_n \to 0 = C(x),$$

so $(A_1, A_2, \ldots)$ converges to C strongly. But $(A_1, A_2, \ldots)$ does not converge to C, or indeed to anything else.

6. Suppose that E, F and G are Banach spaces, that $(A_1, A_2, \ldots)$ is a sequence in $\mathcal{L}(F, G)$ which converges strongly to C and that $(B_1, B_2, \ldots)$ is a sequence in $\mathcal{L}(E, F)$ which converges strongly to D. Show that that the sequence $(A_1 B_1, A_2 B_2, \ldots)$ in $\mathcal{L}(E, G)$ converges strongly to CD.
Hint: Use the Uniform Boundedness Principle.
Solution:

$$A_n B_n - CD = (A_n - C)(B_n - D) + (A_n - C)D + C(B_n - D)$$

so, if $x \in E$,

$$A_n B_n x - CDx = (A_n - C)(B_n - D)x + (A_n - C)D + C(B_n - D)x.$$

Now

$$(B_n - D)x = B_n x - Dx \to 0$$

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[1] Yes, the terminology is horrible. You might reasonably expect based on the rules of English grammar that strong convergence is a stronger condition than ordinary convergence, but it's actually a weaker condition.

as $n \to \infty$, because $B_n \to D$ strongly. But then

$$C(B_n - D)x \to C0 = 0$$

because C is continuous. Also,

$$(A_n - C)Dx \to 0$$

because $A_n \to C$ strongly and $Dx \in F$.
Finally, for each $y \in F$

$$(A_n - C)y$$

converges and so is bounded

$$\sup \|(A_n - C)y\| < \infty.$$

By the Uniform Boundedness Principle then

$$\sup \|A_n - C\| < \infty.$$

Let

$$\gamma > \sup \|A_n - C\| \geq 0.$$

If $\epsilon > 0$ then $\epsilon/\gamma > 0$ so there an N such that if $j > N$ then

$$\|B_n x - Dx\| < \frac{\epsilon}{\gamma}.$$

But then

$$\|(A_n - C)(B_n - D)x\| < \epsilon$$

and so

$$(A_n - C)(B_n - D) \to 0.$$

Combining all these

$$A_n B_n x - CDx \to 0$$

or

$$A_n B_n x \to CDx.$$

That means $A_n B_n \to CD$ strongly.