MA 3421 Assignment 7, Due 15 November 2018

Solutions

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1. Prove the following 'converse' to the spectral theorem:
   If $\{u_1, u_2, \ldots\}$ is an orthonormal set in a Hilbert space E and $(\lambda_1, \lambda_2, \ldots)$ is a sequence in $\mathbf{R}$ with limit 0 and $|\lambda_1| \geq |\lambda_2| \geq \cdots$ then

$$Ax = \sum_{j=1}^{\infty} \lambda_j \, (x|u_j) \, u_j$$

   defines a compact symmetric operator on E.
   Hint: Remember that the limit of a sequence of compact operators is compact.
   Solution: A is clearly linear. Also,

$$(Ax|y) = \left(\sum_{j=1}^{\infty} \lambda_j \, (x|u_j) \, u_j | y \right) = \sum_{j=1}^{\infty} \lambda_j \, (x|u_j) \, (u_j|y)$$

   while

$$(x|Ay) = \left(x | \sum_{j=1}^{\infty} \lambda_j \, (y|u_j) \, u_j \right) = \sum_{j=1}^{\infty} \overline{\lambda_j \, (y|u_j)} \, (x|u_j) = \sum_{j=1}^{\infty} \lambda_j \, (u_j|y) \, (x|u_j) \, ,$$

   so A is symmetric. Define K_n by

$$K_n x = \sum_{j=1}^{n} (x|u_j) \, u_j .$$

   Then K_n is compact because its range is finite dimensional. Also

$$\|(A - K_n)x\|^2 = \sum_{j>n} |\lambda_j|^2 | \, (x|u_j) \, |^2 \leq |\lambda_n|^2 \sum_{j>n} | \, (x|u_j) \, |^2 \leq |\lambda_n|^2 \|x\|^2 .$$

   From this it follows that

$$\|A - K_n\| \leq |\lambda_n\|.$$

Taking limits,

$$\lim_{n\to\infty} \|A - K_n\| = 0$$

and hence

$$\lim_{n\to\infty} K_n = A.$$

Thus A, as the limit of a sequence of compact operators is compact.

2. Show that every compact symmetric operator A on a Hilbert space E can be written in the form

$$A = B - C$$

where B and C are compact positive operators.
Solution: By the spectral theorem we can write

$$Ax = \sum_{j=1}^{\infty} \lambda_j\,(x|u_j)\,u_j$$

with an orthonormal set $\{u_1, u_2, \ldots\}$ and a sequence $(\lambda_1, \lambda_2, \ldots)$ in $\mathbf{R}$ with limit 0 and $|\lambda_1| \geq |\lambda_2| \geq \cdots$. Define J_+ and J_- by

$$J_+ = \{j \in \mathbf{Z}^+ \colon \lambda_j > 0\}, \quad J_- = \{j \in \mathbf{Z}^+ \colon \lambda_j < 0\}$$

and define B and C by

$$Bx = \sum_{j\in J_+} \lambda_j\,(x|u_j)\,u_j, \quad Cx = -\sum_{j\in J_-} \lambda_j\,(x|u_j)\,u_j.$$

Then

$$Ax = Bx - Cx$$

for all $x \in E$. By the preceding problem B and C are compact symmetric operators. Also

$$(Bx|x) = \sum_{j\in J_+} \lambda_j|\,(x|u_j)\,|^2 \quad (Cx|x) = \sum_{j\in J_-} (-\lambda_j)|\,(x|u_j)\,|^2$$

are both sums of non-negative terms, so

$$(Bx|x) \geq 0, \quad (Cx|x) \geq 0$$

and therefore B and C are positive.

3. The left and right unilateral shifts $L, R \in \mathcal{L}(l^2)$ are defined by

$$Lx = (\xi_2, \xi_3, \ldots), \quad Rx = (0, \xi_1, \xi_2, \ldots),$$

where

$$x = (\xi_1, \xi_2, \ldots).$$

(a) Show that neither L nor R is compact.

(b) Show that neither L nor R is symmetric.

(c) Find the eigenvectors and eigenvalues of L.

(d) Find the eigenvectors and eigenvalues of R.

Solution:

(a) Let e_n be the n'th standard basis vector. Then $Le_1 = 0$, $e_{n+1} = e_n$ and $Re_n = e_{n+1}$. $(e_1, e_2, \ldots)$ is a bounded sequence. Neither $(Le_1, Le_2, \ldots)$ nor $(Re_1, Re_2, \ldots)$ has a convergent subsequence.

(b)

$$(Le_1|e_2) = 0 \neq 1 = (e_1|Le_2)$$

and

$$(Re_2|e_1) = 0 \neq 1 = (e_2|Re_1)\,.$$

(c)

$$Lx = \lambda x$$

if and only if

$$\xi_2 = \lambda\xi_1, \quad \xi_3 = \lambda\xi_2, \ldots.$$

This happens if and only if there is an α such that for all n

$$\xi_n = \alpha\lambda^n.$$

For an eigenvector we need $\alpha \neq 0$. Then $x \in l^2$ if and only if

$$|\lambda| < 1.$$

(d)

$$Rx = \lambda x$$

if and only if

$$0 = \lambda\xi_1, \quad \xi_1 = \lambda\xi_2, \ldots.$$

If $\lambda \neq 0$ then there are no non-zero solutions to this. If

$$\lambda = 0$$

then this happens precisely when x is a multiple of e_1.

4. Show that $\{1, \sqrt{2}\cos(\pi t), \sqrt{2}\cos(2\pi t), \ldots\}$ and $\{\sqrt{2}\sin(\pi t), \sqrt{2}\sin(2\pi t), \ldots\}$ are orthonormal bases for $L^2([0,1])$.
Hint: Write them as solutions to a Sturm-Liouville problem.
Solution: The set $\{1, \sqrt{2}\cos(\pi t), \sqrt{2}\cos(2\pi t), \ldots\}$ are solutions to the Sturm-Liouville problem

$$x'' + \lambda x = 0, \quad x'(0) = 0, \quad x'(1) = 0$$

with $\lambda = n^2\pi^2$, $n \in \{0, 1, 2, \ldots\}$. Furthermore, every solution to this Sturm-Liouville problem is a multiple of one included in the list because the differential equation together with the first of the two boundary conditions must be of the form $x(t) = \alpha\cos(\pi\xi t)$ and the second boundary condition then forces ξ to be an integer. There is an orthormal basis consisting of eigenvectors, by the spectral theory argument presented in lecture and in the notes, and we've listed all the eigenvectors, so this is it. For the set $\{\sqrt{2}\sin(\pi t), \sqrt{2}\sin(2\pi t), \ldots\}$ the argument is similar, but now the Sturm-Liouville problem is

$$x'' + \lambda x = 0, \quad x(0) = 0, \quad x(1) = 0,$$

the values of λ are $\lambda = n^2\pi^2$, $n \in \{1, 2, \ldots\}$ and the solutions to the differential equation and first boundary condition are of the form $x(t) = \alpha\sin(\pi\xi t)$.

5. Show that $\{\frac{1}{\sqrt{2}}, \cos(\pi t), \sin(\pi t), \cos(2\pi t), \sin(2\pi t), \ldots\}$ is an orthonormal basis for $L^2([-1, 1])$.
Hint: Use the result of the previous problem. You may find the functions $A\colon L^2([-1, 1]) \to L^2([0, 1]) \oplus L^2([0, 1])$ and $B\colon L^2([0, 1]) \oplus L^2([0, 1]) \to L^2([-1, 1])$ useful where

$$A(x) = (y, z), \quad y(t) = \frac{x(t) + x(-t)}{\sqrt{2}}, \quad z(t) = \frac{x(t) - x(-t)}{\sqrt{2}}$$

and

$$B((y, z)) = x, \quad x(t) = \begin{cases} \frac{y(t)+z(t)}{\sqrt{2}} & \text{if } t > 0, \\ \frac{y(t)-z(t)}{\sqrt{2}} & \text{if } t < 0. \end{cases}$$

Solution: Note that $x(0)$ is left undefined, but $\{0\}$ is of measure zero, so $B((y, z))$ is a well defined element of $L^2([-1, 1])$. Also AB is the identity operator on $L^2([0, 1]) \oplus L^2([0, 1])$ while BA is the identity operator on $L^2([-1, 1])$. In addition, if $Ax = (y, z)$ and $Au = (v, w)$ then

$$(v|y)_{L^2([0,1])} = \int_0^1 v(t)\overline{y(t)}\,dt = \int_0^1 \frac{[u(t) + u(-t)][\overline{x(t)} + \overline{x(-t)}]}{2}\,dt,$$

$$(w|z)_{L^2([0,1])} = \int_0^1 w(t)\overline{z(t)}\,dt = \int_0^1 \frac{[u(t) - u(-t)][\overline{x(t)} - \overline{x(-t)}]}{2}\,dt$$

and

$$\begin{aligned}
(Au|Ax)_{L^2([0,1])\oplus L^2([0,1])} &= ((v,y)|(w,z))_{L^2([0,1])\oplus L^2([0,1])} \\
&= (v|y)_{L^2([0,1])} + (w|z)_{L^2([0,1])} \\
&= \int_0^1 [u(t)\overline{x(t)} + u(-t)\overline{x(-t)}]\,dt \\
&= \int_0^1 u(t)\overline{x(t)}\,dt + \int_0^1 [u(-t)\overline{x(-t)}]\,dt \\
&= \int_0^1 u(t)\overline{x(t)}\,dt + \int_{-1}^0 [u(t)\overline{x(t)}]\,dt \\
&= \int_{-1}^1 u(t)\overline{x(t)}\,dt \\
&= (u|x)_{L^2([-1,1])}.
\end{aligned}$$

Since the definition of an orthonormal bases involves only the inner product structure and the associated norm, $\{1/\sqrt{2}, \cos(\pi t), \sin(\pi t), \ldots\}$ is an orthonormal basis for $L^2([-1,1])$ if and only if its image under A is an orthonormal basis for $L^2([0,1]) \oplus L^2([0,1])$. This image is just

$$\{(1,0), (\sqrt{2}\cos(\pi t), 0), (0, \sqrt{2}\sin(\pi t))(\sqrt{2}\cos(2\pi t), 0), (0, \sqrt{2}\sin(2\pi t)), \ldots\},$$

which is the union of

$$\{(1,0), (\sqrt{2}\cos(\pi t), 0), (\sqrt{2}\cos(2\pi t), 0), \ldots\}$$

and

$$\{(0, \sqrt{2}\sin(\pi t)), (0, \sqrt{2}\sin(2\pi t)), \ldots\}.$$

Since $\{1, \sqrt{2}\cos(\pi t), \sqrt{2}\cos(2\pi t), \ldots\}$ and $\{\sqrt{2}\sin(\pi t), \sqrt{2}\sin(2\pi t), \ldots\}$ are orthonormal bases for $L^2([0,1])$ is follows that the union above is indeed an orthonormal basis for $L^2([0,1]) \oplus L^2([0,1])$.