MA 3421 Assignment 6, Due 8 November 2018

Solutions

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1. Suppose S is a set, not necessarily finite or countable, $f: S \to \mathbf{R}$ is a function, $\mu, \nu \in \mathbf{R}$ and, for all finite subsets $\subseteq S$ we have

$$\mu \leq \sum_{u \in S} f(u) \leq \nu.$$

Show that the set

$$N = \{u \in S: f(u) \neq 0\}$$

is at most countable.
Note: This is a generalisation of an argument given both in the notes and in lecture.
Solution: Set

$$S_{+\delta} = \{u \in S: f(u) \geq \delta\}$$

and

$$S_{-\delta} = \{u \in S: f(u) \leq -\delta\}$$

for $\delta > 0$. If $T \subseteq S_{+\delta}$ is finite then

$$\sum_{u \in T} f(u) \geq n\delta$$

where n is the number of elements of T. It follows that

$$n \leq \mu/\delta.$$

Since this holds for all finite subsets T it follows that $S_{+\delta}$ itself has at most that many elements. Similarly, if $T \subseteq S_{-\delta}$ is finite then

$$\sum_{u \in T} f(u) \leq -n\delta$$

where n is the number of elements of T. It follows that

$$n \leq -\nu/\delta.$$

Since this holds for all finite subsets T it follows that $S_{+\delta}$ itself has at most that many elements. So

$$N = \cup S_{\pm 1/n}$$

as a countable union of finite sets is at most countable.

2. The dual to a normed space E is, by definition, $E' = \mathcal{L}(E, \mathbf{K})$. This is a normed space, because $\mathbf{K}$ is a normed space and $\mathcal{L}(E, F)$ is a normed space for any normed spaces E and F. The double dual is the dual of the dual, $E'' = \mathcal{L}(E', \mathbf{K}) = \mathcal{L}(\mathcal{L}(E, \mathbf{K}), \mathbf{K})$. Show that E' and E'' are in fact Banach spaces.
Solution: We saw earlier that $\mathcal{L}(E, F)$ is a Banach space whenever F is. $\mathbf{K}$ is a Banach space, so E' is a Banach space. Applying the xame argument with E' in place of E shows that E'' is a Banach space.

3. Define $J(x)$, a function from E' to $\mathbf{K}$, by

$$(J(x))(f) = f(x)$$

for all $x \in E$, $f \in E'$. Show the following:

(a) $J(x)\colon E' \to \mathbf{K}$ is linear and bounded, so $J(x) \in E''$.

(b) $J\colon E \to E''$ is linear and bounded, so $J \in \mathcal{L}(E, E'')$.

(c) If E is an inner product space then J is injective.

Solution:

(a)

$$\begin{aligned} (J(x))(\alpha f + \beta g) &= (\alpha f + \beta g)(x) \\ &= \alpha f(x) + \beta g(x) \\ &= \alpha (J(x))(f) + \beta (J(x))(g) \end{aligned}$$

for all $\alpha, \beta \in \mathbf{K}$ and $f, g \in E'$, so $J(x)$ is linear.

$$\|(J(x))(f)\|_{\mathbf{K}} = \|f(x)\|_{\mathbf{K}} \leq \|f\|_{E'} \|x\|_E = \|x\|_E \|f\|_{E'}$$

for all $f \in E'$ so $J(x)$ is bounded, $J(x) \in E''$ and

$$\|J(x)\|_{E''} \leq \|x\|_E.$$

(b)

$$\begin{aligned} (J(\alpha x + \beta y))(f) &= f(\alpha x + \beta y) \\ &= \alpha f(x) + \beta f(y) \\ &= \alpha (J(x))(f) + \beta (J(y))(f) \end{aligned}$$

for all $\alpha, \beta \in \mathbf{K}$, $x, y \in E$ and $f \in E'$, so

$$J(\alpha x + \beta y) = \alpha J(x) + \beta J(y)$$

for all $\alpha, \beta \in \mathbf{K}$, $x, y \in E$. In other words, J is linear. Also

$$\|J(x)\|_{E''} \leq \|x\|_E.$$

for all $x \in E$, so J is bounded and

$$\|J\|_{\mathcal{L}(E,E'')} \leq 1.$$

(c) Suppose

$$Jx = 0.$$

Then

$$(Jx)(f) = 0$$

for all $f \in E'$. In other words

$$f(x) = 0$$

for all f in E'. But

$$f(y) = (y|x)$$

defines an element $f \in E'$, so

$$(x|x) = 0$$

and hence

$$x = 0.$$

4. If E is a Hilbert space then it's a normed space and so E' is a Banach space. Show that it is, in fact, a Hilbert space, i.e. that the norm on E' is induced by an inner product on E'.
Note: Use the Riesz Representation Theorem.
Solution: Suppose $f, g \in E'$. By the Riesz Representation Theorem there are $w, z \in E$ such that

$$f(x) = (x|w), \quad g(y) = (y|z)$$

for all $x, y \in E$. These are unique, because if, for example

$$f(x) = (x|u)$$

for all $x \in E$ then

$$(x|u - w) = (x|u) - (x|w) = f(x) - f(x) = 0$$

for all $x \in E$, including $x = u - w$. Thus

$$\|u - w\|^2 = (u - w|u - w) = 0.$$

So it makes sense to define a function $q\colon E' \times E' \to \mathbf{K}$ by

$$q(f, g) = (z|w).$$

We now show that this is an inner product. If $h, k \in E'$ and $f = h + k$ then, choosing u, v in E such that

$$h(x) = (x|u)\,, \quad k(x) = (x|v)$$

for all $x \in E$ we have

$$f(x) = h(x) + k(x) = (x|u) + (x|v) = (x|u + v) = (x|w)$$

for all $x \in E$, where

$$w = u + v.$$

So

$$q(h + k, g) = q(f, g) = (z|u + v) = (z|u) + (z|v) = q(h, g) + q(k, g).$$

If $\alpha \in \mathbf{K}$ then

$$\alpha f(x) = \alpha\,(x|w) = (x|\overline{\alpha}w)$$

for all $x \in E$ so

$$q(\alpha f, g) = (z|\overline{\alpha}w) = \alpha\,(z|w) = \alpha q(f, g).$$

Also

$$q(f, g) = (z|w) = \overline{(w|x)} = \overline{q(g, f)},$$

$$q(f, f) = (w|w) \geq 0,$$

and $q(f, f) = 0$ only if $w = 0$, in which case $f = 0$. So q is an inner product on E'. We have

$$q(f, f) = (w|w) = \|w\|^2 = \|f\|^2,$$

so

$$\|f\| = \sqrt{q(f, f)}.$$

In other words, the norm on E' is the one induced by the inner product q.

5. Suppose that E and F are normed spaces and $A\colon E \to F$ linear. and that the image of every bounded sequence in E has a convergent subsequence, i.e. that for every bounded sequence $(x_1, x_2, \ldots)$ in E the sequence $(Ax_1, Ax_2, \ldots)$ in F has a convergent subsequence. Show that A is compact.
Note: Your first task is to figure out why this is not just the definition of compactness.
Solution: The only part which is missing from the definition of compactness is the boundedness of A, which we can prove by contradiction. If A is not bounded then for each $n \in \mathbf{Z}^+$ we have that n is not a bound, i.e. that there is an $x_n \in E$ with

$$\|Ax_n\| > n\|x_n\|.$$

Let

$$y_n = \frac{x_n}{\|x_n\|}.$$

Then $\|y_n\| = 1$ and $\|Ay_n\| > n$. The sequence $(y_1, y_2, \ldots)$ is bounded but the sequence $(Ay_1, Ay_2, \ldots)$ is not. Neither is any of its subsequences, so none of them is convergent, contradicting the compactness of A.

6. Suppose $s \in \mathbf{K}$ and $\operatorname{Re} s \geq 0$. If $x = (\xi_1, \xi_2, \ldots) \in l^2$ then

$$A_n x = (1^{-s}\xi_1, 2^{-s}, \ldots, n^{-s}\xi_n, 0, 0, \ldots),$$

$$Bx = (1^{-s}\xi_1, 2^{-s}, \ldots, n^{-s}\xi_n, (n+1)^{-s}\xi_{n+1}, \ldots),$$

belong to l^2 and define operators $A_n, B \in \mathcal{L}(l^2)$.

(a) Show that

$$\lim_{n\to\infty} A_n = B$$

if $\operatorname{Re} s > 0$.

(b) Show that A_n is compact if $\operatorname{Re} s \geq 0$.
Hint: Write A_n as a product of operators with finite dimensional domain or range.

(c) Show that B is compact if $\operatorname{Re} s > 0$.

(d) Show that B is symmetric if and only if s is real.

Solution:

(a)

$$(B - A_n)x = (0, 0, \ldots, (n+1)^{-s}\xi_{n+1}, (n+2)^{-s}\xi_{n+2}, \ldots),$$

and

$$\begin{aligned}\|(A_n - B)x\|^2 &= ((B - A_n)x|(B - A_n)x) \\ &= (n+1)^{-2\operatorname{Re} s}|\xi_{n+1}|^2 + (n+2)^{-2\operatorname{Re} s}|\xi_{n+2}|^2 + \ldots\end{aligned}$$

so

$$\|(A_n - B)x\|^2 \leq (n+1)^{-2\operatorname{Re} s}\|x\|^2,$$

$$\|(A_n - B)x\| \leq (n+1)^{-\operatorname{Re} s}\|x\|,$$

and, since x was arbitrary,

$$\|(A_n - B)\| \leq (n+1)^{-\operatorname{Re} s}.$$

The right hand side tends to zero as $n \to \infty$, so

$$\lim_{n\to\infty} \|A_n - B\| = 0$$

and hence

$$\lim_{n\to\infty} A_n = B.$$

(b) Write

$$A_n = R_n S_n T_n$$

where

$$R_n(\xi_1, \ldots, \xi_n) = (\xi_1, \ldots, \xi_n, 0, 0, 0),$$
$$S_n(\xi_1, \ldots, \xi_n) = (1^{-s}\xi_1, \ldots, n^{-s}\xi_n),$$

and

$$T_n(\xi_1, \xi_2, \ldots) = (\xi_1, \ldots, \xi_n)$$

are operators in $\mathcal{L}(l^2(n), l^2)$, $\mathcal{L}(l^2(n), l^2(n))$ and $\mathcal{L}(l^2, l^2(n))$ respectively. All of them are compact, since $\mathcal{L}(E, F)$ is compact if either E or F is finite dimensional. The product $R_n S_n T_n$ is therefore compact. It would in fact suffice if any one of the factors were compact.

(c) Using the two preceding parts B is a limit of compact operators and hence is compact.

(d) If s is real, $x = (\xi_1, \xi_2, \ldots) \in l^2$ and $y = (\eta_1, \eta_2, \ldots) \in l^2$ then then

$$(Bx|y) = \sum_{j=1}^{\infty} j^{-s}\xi_j\overline{\eta_j} = \sum_{j=1}^{\infty} \xi_j\overline{j^{-s}\eta_j} = (x|By)\,,$$

so B is symmetric. Conversely, suppose B is symmetric.

$$(Be_j|e_j) = j^{-s} = \exp(-s\log j)$$

and

$$(e_j|Be_j) = \overline{j^{-s}} = \exp(-\overline{s}\log j),$$

with $\{e_1, e_2, \ldots\}$ being the standard basis for l^2, B is symmetric so the two are equal, which implies

$$(s - \overline{s})\log j = 2\pi i m$$

or

$$\frac{\operatorname{Im} s}{\pi}\log j = m$$

for some $m \in \mathbf{Z}$. We apply this with $j = p$ and $j = q$ where p and q are distinct primes.

$$\frac{\operatorname{Im} s}{\pi}\log p = m_p, \quad \frac{\operatorname{Im} s}{\pi}\log q = m_q$$

Then

$$m_p \log q = m_q \log p$$

and, exponentiating

$$q^{m_p} = p^{m_q}.$$

Because of unique factorisation

$$m_p = m_q = 0$$

and hence, because $\log p \neq 0$,

$$\frac{\operatorname{Im} s}{\pi} = 0.$$