MA 3421 Assignment 5, Due 1 November 2018

Solutions

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1. It was shown in lecture that if $x_1, x_2, \ldots$ are pairwise orthogonal in an inner product space E and
$$\sum_{j=1}^{\infty} x_j$$
converges then
$$\left\| \sum_{j=1}^{\infty} x_j \right\|^2 = \sum_{j=1}^{\infty} \|x_j\|^2 .$$
Show that if $x_1, x_2, \ldots$ are pairwise orthogonal in a Hilbert space E and
$$\sum_{j=1}^{\infty} \|x_j\|^2$$
converges then
$$\sum_{j=1}^{\infty} x_j$$
converges and
$$\left\| \sum_{j=1}^{\infty} x_j \right\|^2 = \sum_{j=1}^{\infty} \|x_j\|^2 .$$
Solution: Since Hilbert spaces are complete it suffices to show that the sequence of partial sums
$$s_n = \sum_{j=1}^{n} x_j$$
is Cauchy. Suppose $\epsilon > 0$. If $k < l$ then
$$\|s_l - s_k\|^2 = \left\| \sum_{j=k+1}^{l} x_j \right\|^2 = \sum_{j=k+1}^{l} \|x_j\|^2 = \sigma_l - \sigma_k = |\sigma_l - \sigma_k|$$

where

$$\sigma_n = \sum_{j=1}^{n} \|x_j\|^2.$$

In the second step we used identity for finite sums of pairwise orthogonal vectors, proved in the lecture notes. Similarly, if $k > l$ we have

$$\|s_l - s_k\|^2 = \left\| \sum_{j=l+1}^{k} x_j \right\|^2 = \sum_{j=l+1}^{k} \|x_j\|^2 = \sigma_k - \sigma_l = |\sigma_l - \sigma_k|.$$

In either case

$$\|s_l - s_k\|^2 = |\sigma_l - \sigma_k|$$

and this holds trivially also if $k = l$. By assumption the sequence $(\sigma_1, \sigma_2, \ldots)$ is convergent and hence Cauchy. Also $\epsilon^2 > 0$ so there is an N such that $k, l > N$ implies

$$|\sigma_l - \sigma_k| < \epsilon^2$$

and hence

$$\|s_l - s_k\| < \epsilon.$$

Hence $(s_1, s_2, \ldots)$ is Cauchy and hence, since E is Hilbert space, convergent. In other words,

$$\sum_{j=1}^{\infty} x_j$$

converges. Once we know this the identity

$$\left\| \sum_{j=1}^{\infty} x_j \right\| = \sum_{j=1}^{\infty} \|x_j\|^2$$

follows.

2. The functions

$$x_j(t) = t^j \exp(-t^2/2)$$

in $L^2(\mathbf{R})$ are linearly independent. Use them to find an orthonormal set via Gram-Schmidt.
Note: There are, of course, infinitely many of them, so you're not going to be able to list them all. Just find the ones of degree less than 4.
Solution: We need the integral

$$\int_{-\infty}^{\infty} t^j \exp(-t^2)\, dt = \Gamma\left(\frac{j+1}{2}\right) = \begin{cases} \sqrt{\pi} & \text{if } j = 0, \\ 0 & \text{if } j = 2k+1, \\ \frac{1}{2^{2k+1}} \frac{(2k+1)!}{k!} \sqrt{\pi} & \text{if } j = 2k+2. \end{cases}$$

For simplicity we start with index 0 rather than index 1. Then

$$x_0(t) = t^0 \exp(-t^2/2) = \exp(-t^2/2),$$

$$y_0(t) = x_0(t) = \exp(-t^2/2),$$

$$\|y_0\|^2 = (y_0|y_0) = \int_{-\infty}^{\infty} \exp(-t^2)\,dt = \sqrt{\pi},$$

$$u_0(t) = y_0(t)/\|y_0\| = \pi^{-1/4}\exp(-t^2/2),$$

$$x_1(t) = t^1\exp(-t^2/2) = t\exp(-t^2/2),$$

$$\begin{aligned} y_1(t) &= x_1(t) - (x_1|u_0)\,u_0(t) \\ &= t\exp(-t^2/2) - \left(\int_{-\infty}^{\infty} \pi^{-1/4} t\exp(-t^2)\,dt\right)\pi^{-1/4}\exp(-t^2/2) \\ &= t\exp(-t^2/2) - 0, = t\exp(-t^2/2), \end{aligned}$$

$$\|y_1\|^2 = (y_1|y_1) = \int_{-\infty}^{\infty} t^2\exp(-t^2)\,dt = \frac{1}{2}\sqrt{\pi},$$

$$u_1(t) = y_1(t)/\|y_1\| = 2^{1/2}\pi^{-1/4} t\exp(-t^2/2),$$

$$x_2(t) = t^2\exp(-t^2/2)$$

$$\begin{aligned} y_2(t) &= x_2(t) - (x_2|u_0)\,u_0(t) - (x_2|u_1)\,u_1(t) \\ &= t^2\exp(-t^2/2) - \left(\int_{-\infty}^{\infty} \pi^{-1/4} t^2\exp(-t^2)\,dt\right)\pi^{-1/4}\exp(-t^2/2) \\ &\quad - \left(\int_{-\infty}^{\infty} 2^{1/2}\pi^{-1/4} t^3\exp(-t^2)\,dt\right)2^{1/2}\pi^{-1/4} t\exp(-t^2/2) \\ &= t^2\exp(-t^2/2) - \frac{1}{2}\exp(-t^2/2) - 0 = (t^2 - 1/2)\exp(-t^2/2), \end{aligned}$$

$$\|y_2\|^2 = (y_2|y_2) = \int_{-\infty}^{\infty} (t^2 - 1/2)^2\exp(-t^2)\,dt = \frac{1}{2}\sqrt{\pi},$$

$$u_2(t) = y_2(t)/\|y_2\| = 2^{1/2}\pi^{-1/4}(t^2 - 1/2)\exp(-t^2/2),$$

$$x_3(t) = t^3\exp(-t^2/2)$$

$$\begin{aligned} y_3(t) &= x_3(t) - (x_3|u_0)\,u_0(t) - (x_3|u_1)\,u_1(t) - (x_3|u_2)\,u_2(t) \\ &= t^3\exp(-t^2/2) - \left(\int_{-\infty}^{\infty} \pi^{-1/4} t^3\exp(-t^2)\,dt\right)\pi^{-1/4}\exp(-t^2/2) \\ &\quad - \left(\int_{-\infty}^{\infty} 2^{1/2}\pi^{-1/4} t^4\exp(-t^2)\,dt\right)2^{1/2}\pi^{-1/4} t\exp(-t^2/2) \\ &\quad - \left(\int_{-\infty}^{\infty} 2^{1/2}\pi^{-1/4}(t^5 - t^3/2)\exp(-t^2)\,dt\right)2^{1/2}\pi^{-1/4}(t^2 - 1/2)\exp(-t^2/2) \\ &= t^3\exp(-t^2/2) - 0 - \frac{3}{2}t\exp(-t^2/2) - 0 = (t^3 - 3t/2)\exp(-t^2/2), \end{aligned}$$

$$\|y_3\|^2 = (y_3|y_3) = \int_{-\infty}^{\infty} (t^3 - 3t/2)^2\exp(-t^2)\,dt = \frac{3}{4}\sqrt{\pi},$$

$$u_3(t) = y_3(t)/\|y_3\| = 2\cdot 3^{-1/2}\pi^{-1/4}(t^3 - 3t/2)\exp(-t^2/2), \ldots$$

3. Suppose $\{x_1, \ldots, x_n\} \subset E$ where E is an inner product space and let A be the matrix whose j'th row, k'th column is

$$\alpha_{j,k} = (x_j | x_k) \,.$$

   (a) Show that A is Hermitian and positive semi-definite.

   (b) Show that $\{x_1, \ldots, x_n\}$ is linearly dependent if and only if $\det A = 0$.

Solution: That A is Hermitian is clear, since

$$\alpha_{k,j} = (x_k | x_j) = \overline{x_j} x_k = \overline{\alpha_{j,k}}.$$

If

$$z = \sum_{j=1}^{n} \beta_j x_j$$

then

$$(z|z) = \sum_{j=1}^{n} \sum_{k=1}^{n} \beta_j \overline{\beta_k} \, (x_j | x_k) = \sum_{j=1}^{n} \sum_{k=1}^{n} \beta_j \overline{\beta_k} \alpha_{j,k} = bAb^*$$

where the j'th column of the row vector b is β_j and the star denotes conjugate transpose. So for any row vector b we have

$$bAb^* \geq 0$$

because the left hand side is $(z|z)$ for some $z \in E$. This shows that A is positive semi-definite.

If $\det A = 0$ then there is a non-zero row vector b such that

$$bAb^* = 0$$

Defining $\beta_1, \ldots, \beta_n$ and z as above we find that $(z|z) = 0$ and hence $z = 0$ and

$$\sum_{j=1}^{n} \beta_j x_j = 0,$$

so $\{x_1, \ldots, x_n\}$ is a linearly dependent set.

Conversely, suppose $\{x_1, \ldots, x_n\}$ is a linearly dependent set, i.e. that there are $\beta_1, \ldots, \beta_n$, not all of which are zero, such that

$$\sum_{j=1}^{n} \beta_j x_j = 0,$$

In that case, forming b and z as above, $b \neq 0$ and

$$bAb^* = 0$$

so b^* is a non-zero vector in the null space of A, which must then have determinant zero.

4. Suppose that $P \in \mathcal{L}(E)$,
$$P^2 = P$$
and, for all $x, y \in E$,
$$(Px|y) = (x|Py) .$$
Show that there is a closed subspace F such that P is the orthogonal projection onto F.
Solution: Let F be the null space of $I-P$. F is certainly closed because it's the inverse image of the closed set $\{0\}$ under the continuous function $I-P$. If $x \in E$ then
$$x = u + v$$
where
$$u = Px, \quad v = (I-P)x.$$
Now $(I-P)u = (P-P^2)x = 0$, so
$$u \in F.$$
If $y \in F$ then
$$(I-P)y = 0$$
and hence
$$Py = Iy = y.$$
Then
$$(v|y) = (x|y)-(u|y) = (x|y)-(Px|y) = (x|y)-(x|Py) = (x|y)-(x|y) = 0.$$
So v is orthogonal to y. Since $y \in F$ was arbitrary it follows that
$$v \in F^{\perp}.$$
The decomposition $x = u+v$ is therefore the one which was used to define the orthogonal projection.

5. The proof given in the notes that any orthonormal set is a subset of a maximal orthonormal set uses Zorn's Lemma and so is non-constructive. Give a constructive proof in $l^2(n)$, i.e. describe an algorithm which, given an orthonormal set $\{u_1, \ldots, u_k\} \subseteq l^2$, constructs $u_{k+1}, \ldots, u_n$ such that $\{u_1, \ldots, u_n\}$ is an orthonormal set.
Note: You can assume for purposes of this problem that you have a way of doing exact arithmetic in $\mathbf{K}$. If you're feeling adventurous you can try to give an algorithm which has some hope of working in real life.
Hint: Try to adapt the Gram-Schmidt algorithm.
Solution: Assuming we have an orthonormal set $\{u_1, \ldots, u_l\}$ with $l < n$ we can form the vectors
$$y_{l,m} = e_m - \sum_{j=1}^{n} (e_m|u_j)\, u_j$$

for $1 \le m \le n$, where e_m is the m'th standard basis vector. Since $l < n$ at least one of these is non-zero, since otherwise we can write the standard basis as linear combinations of the smaller set $\{u_1, \ldots, u_l\}$. Choose m_l to be the m which gives the largest value of $(y_m|y_m)$ and set

$$u_{l+1} = \frac{y_{l,m_l}}{\|y_{l,m_l}\|}.$$

We start, of course, with $l = k$ and in $n - k$ steps we are done. The resulting set $\{u_1, \ldots, u_n\}$ is certainly an orthonormal set, because it's the result of applying Gram-Schmidt to $\{u_1, \ldots, u_k, e_{m_k}, \ldots, e_{m_{n-1}}\}$. The set is maximal because if there were a larger one then we'd have a set of more than n linearly independent vectors in $\mathbf{K}^n$.