MA 3421 Assignment 4, Due 18 October 2018

Solutions

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1. Suppose $P \in \mathcal{L}(E)$ is a projection on a normed space E, i.e. that $P^2 = P$. Suppose also that $P \neq 0$. Show that $\|P\| \geq 1$.
   Hint: Proof by contradiction is your friend.
   Solution:
   $$\|P\| = \|P^2\| \leq \|P\|\|P\|$$
   so either $\|P\| = 0$, in which case $P = 0$, or
   $$1 \leq \|P\|.$$
   Alternatively, if $\|P\| < 1$ then $I - P$ is invertible. But then
   $$P = PI = P(I-P)(I-P)^{-1} = (P-P^2)(I-P)^{-1} = 0(I-P)^{-1} = 0$$
   if $P - P^2 = 0$.

2. Suppose that
   $$\lim_{n\to\infty} x_n = y$$
   in a Banach algebra E with identity element e. If there is no n for which x_n is invertible then show that y is not invertible.
   Solution: Suppose y is invertible. Take
   $$\epsilon = \frac{1}{\|y^{-1}\|}.$$
   Then $\epsilon > 0$ so there is an N such that $n > N$ implies
   $$\|x_n - y\| < \epsilon.$$
   But this implies that x_n is invertible.

3. Suppose that $(f_1, f_2, \ldots)$ is a sequence of bounded functions on $\mathbf{R}$ converging uniformly to g. Suppose also that

$$\inf_{r \in \mathbf{R}} |f_n(t)| = 0$$

for each n. Show that

$$\inf_{r \in \mathbf{R}} |g(t)| = 0.$$

Solution: The set of bounded functions on $\mathbf{R}$ is a Banach space with multiplication defined pointwise and

$$\|h\| = \sup_{t \in \mathbf{R}} |h(t)|$$

as norm. The multiplicative inverse to h, if it exists, must be

$$h^{-1}(t) = 1/h(t)$$

since this is the only function satisfying $h^{-1}h = 1$. But

$$\|h^{-1}\| = \sup_{t \in \mathbf{R}} |h^{-1}(t)| = 1/ \inf_{t \in \mathbf{R}} |h(t)| < \infty$$

if and only if

$$\inf_{t \in \mathbf{R}} |h(t)| > 0.$$

So our hypothesis is that $f_n \to g$ and none of the f_n are invertible. It follows from the preceding problem that g is also not invertible, i.e. that

$$\inf_{r \in \mathbf{R}} |g(t)| = 0.$$

4. Show that in any inner product space

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

for all vectors x, y.
Solution:

$$\|x + y\|^2 = (x + y|x + y) = (x|x) + (x|y) + (y|x) + (y|y)$$

and

$$\|x - y\|^2 = (x - y|x - y) = (x|x) - (x|y) - (y|x) + (y|y)$$

so

$$\|x + y\|^2 + \|x - y\|^2 = 2\,(x|x) + 2\,(y|y) = 2\|x\|^2 + 2\|y\|^2.$$

5. Give an example of a normed space E and vectors $x, y \in E$ such that the equation above does not hold.
Solution: Almost anything will work. For example, $E = l^1(2)$,

$$x = (1, 0), \quad y = (0, 1)$$

$$x + y = (1, 1), \quad x - y = (1, -1)$$

$$\|x\| = \|y\| = 1, \quad \|x + y\| = \|x - y\| = 2,$$

$$2^2 + 2^2 \neq 2 \cdot 1^2 + 2 \cdot 1^2.$$

6. Suppose that the *real* vector space E is an inner product space. Show that

$$(x|y) = \left\| \frac{x+y}{2} \right\|^2 - \left\| \frac{x-y}{2} \right\|^2.$$

Solution:

$$\left\| \frac{x+y}{2} \right\|^2 = \left(\frac{x+y}{2} | \frac{x+y}{2} \right) = \frac{1}{4}(x|x) + \frac{1}{4}(x|y) + \frac{1}{4}(y|x) + \frac{1}{4}(y|y)$$

and

$$\left\| \frac{x-y}{2} \right\|^2 = \left(\frac{x-y}{2} | \frac{x-y}{2} \right) = \frac{1}{4}(x|x) - \frac{1}{4}(x|y) - \frac{1}{4}(y|x) + \frac{1}{4}(y|y)$$

so

$$\left\| \frac{x+y}{2} \right\|^2 - \left\| \frac{x-y}{2} \right\|^2 = \frac{1}{2}(x|y) + \frac{1}{2}(y|x) = \frac{1}{2}(x|y) + \frac{1}{2}\overline{(x|y)}.$$

Since we've assumed that our vector space is real

$$\overline{(x|y)} = (x|y)$$

and

$$\left\| \frac{x+y}{2} \right\|^2 - \left\| \frac{x-y}{2} \right\|^2 = (x|y).$$