MA 3421 Assignment 3, Due 4 October 2018

Solutions

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1. Recall that norms $r, s\colon E \to \mathbf{R}$ are called equivalent if there are $\mu, \nu \in \mathbf{R}^+$ such that, for all $x \in E$,

$$\mu r(x) \leq s(x) \leq \nu r(x).$$

Show that equivalence of norms is, as the name suggests, an equivalence relation.
Solution: To show that equivalence of norms is an equivalence relation we need to show three things:

(a) If p is a norm then p is equivalent to p.

(b) If p and q are norms and p is equivalent to q then q is equivalent to p.

(c) If p, q and r are norms and p is equivalent to q and q is equivalent to r then p is equivalent to r.

The first is clear, since

$$1p(x) \leq p(x) \leq 1p(x)$$

for all x. For the second, we observe that if

$$\mu p(x) \leq q(x) \leq \nu p(x)$$

for all x then

$$\frac{1}{\nu} q(x) \leq p(x) \leq \frac{1}{\mu} q(x)$$

for all x. For the third we observe that if

$$\mu p(x) \leq q(x) \leq \nu p(x)$$

and

$$\mu' q(x) \leq r(x) \leq \nu' q(x)$$

for all x then

$$\mu\mu' p(x) \leq r(x) \leq \nu\nu' p(x).$$

2. Show that the $l^p(n)$ and $l^q(n)$ norms on $\mathbf{K}^n$ are equivalent for any $1 \leq p, q < \infty$. Try to obtain the sharpest values for μ and ν possible.
Solution: If we don't care about the constants then the easiest way to do this is to say that both the $l^p(n)$ norm and the $l^q(n)$ norm are, as shown in Section 1.6 of the lecture notes, equivalent to the $l^1(n)$ norm and then use the result of the previous problem.

If we want sharp constants then we need to do more work. Let $x = (\xi_1, \ldots, \xi_n)$. Suppose first that $p < q$.

We have

$$0 \leq \frac{|\xi_j|^p}{\sum_{k=1}^n |\xi_k|^p} \leq 1$$

for all j. We know that $\tau^{p/q} \leq \tau$ for $\tau \in [0, 1]$, so

$$\left(\frac{|\xi_j|^p}{\sum_{k=1}^n |\xi_k|^p} \right)^{q/p} \leq \left(\frac{|\xi_j|^p}{\sum_{k=1}^n |\xi_k|^p} \right)$$

or

$$\frac{|\xi_j|^q}{\left(\sum_{k=1}^n |\xi_k|^p\right)^{q/p}} \leq \frac{|\xi_j|^p}{\sum_{k=1}^n |\xi_k|^p}.$$

We then sum over j,

$$\frac{\sum_{j=1}^n |\xi_j|^q}{\left(\sum_{k=1}^n |\xi_k|^p\right)^{q/p}} \leq \frac{\sum_{j=1}^n |\xi_j|^p}{\sum_{k=1}^n |\xi_k|^p} = 1,$$

from which it follows that

$$\|x\|_q^q = \sum_{j=1}^n |\xi_j|^q \leq \left(\sum_{k=1}^n |\xi_j|^p \right)^{q/p} = \|x\|_p^q.$$

Taking q'th roots,

$$\|x\|_q \leq \|x\|_p.$$

The proof above fails for $x = 0$, but the inequality obviously holds in that case anyway.

The function $\varphi(\tau) = |\tau|^{q/p}$ is strictly convex. Applying Jensen's inequality,

$$\left| \sum_{j=1}^n \frac{1}{n} |\xi_j|^p \right|^{q/p} \leq \sum_{j=1}^n \frac{1}{n} \left| |\xi_j|^p \right|^{q/p} = \sum_{j=1}^n \frac{1}{n} |\xi_j|^q$$

or

$$n^{-q/p} \|x\|_p^q \leq n^{-1} \|x\|_q^q$$

from which we get

$$\|x\|_p \leq n^{1/p - 1/q} \|x\|_q.$$

Alternately, using Hölder's inequality,

$$\begin{aligned}\|x\|_p^p = \sum_{j=1}^n |\xi_j|^p = \sum_{j=1}^n 1|\xi_j|^p &\le \left(\sum_{j=1}^n 1^{p/(q-p)}\right)^{(q-p)/q} \left(\sum_{j=1}^n (|\xi_j|^p)^{q/p}\right)^{q/p} \\ &= n^{(q-p)/q} \left(\sum_{j=1}^n |\xi_j|^q\right)^{q/p} = n^{(q-p)/q}\|x\|_q^p\end{aligned}$$

and therefore

$$\|x\|_p \le n^{1/p-1/q}\|x\|_q.$$

So for $p < q$ we have

$$\|x\|_q < \|x\|_p \le n^{1/p-1/q}\|x\|_q.$$

Similarly, swapping the roles of p and q, we have

$$\|x\|_p < \|x\|_q \le n^{1/q-1/p}\|x\|_p$$

and hence

$$n^{1/p-1/q}\|x\|_q \le \|x\|_p \le \|x\|_q$$

when $q < p$. Combining these,

$$\min\left(1, n^{1/p-1/q}\right)\|x\|_q \le \|x\|_p \le \max\left(1, n^{1/p-1/q}\right)\|x\|_q$$

for $p \ne q$. But this holds trivially for $p = q$ as well.

3. Suppose F is a subspace of a Banach space E. Show that F, with the norm inherited from E, is a normed space and that it is a Banach space if and only if it is closed.
Solution: A norm on E is a function $p\colon E \to \mathbf{R}$ such that

   (a) for all $x \in E$, $p(x) \ge 0$ and $p(x) > 0$ unless $x = 0$,
   (b) for all $\alpha \in \mathbf{K}$ and $x \in E$, $p(\alpha x) = |\alpha| p(x)$, and
   (c) for all $x, y \in E$, $p(x, y) \le p(x) + p(y)$.

   Since F is a subset of E all of these are true for $x, y \in F$, so p is a norm on F.

   Suppose that F is a Banach space and that $(x_1, x_2, \ldots)$ is a convergent sequence in E. Then it's a Cauchy sequence in E and hence a Cauchy sequence in F. But F is complete, so the sequence converges in F. So the limit of any sequence in F lies in F. In other words, F is closed.

   Suppose that F is closed and that $(x_1, x_2, \ldots)$ is a Cauchy sequence in F. Then it's also Cauchy sequence in E and hence, because E is complete, a convergent sequence in E. But F is a closed subset of E, so the limit of a convergent sequence in F is also in F. Thus the sequence converges in F. Thus every Cauchy sequence in F converges in F, so F is complete, and thus a Banach space.

4. Suppose F is a subspace of a Banach space E. Define $p \colon E/F \to \mathbf{R}$ by

$$p(X) = \inf_{x \in X} \|x\|.$$

Show that p is a seminorm on E/F and that is a norm if and only if F is closed.

Solution: The infimum of a set of non-negative real numbers is non-negative, so

$$p(X) > 0$$

for all $X \in E/F$. Also

$$p(\alpha X) = \inf_{y \in \alpha X} \|y\| = \inf_{x \in X} \|\alpha x\| = \inf_{x \in X} |\alpha| \|x\| = |\alpha| \inf_{x \in X} \|x\| = |\alpha| p(X).$$

Now

$$p(X + Y) = \inf_{z \in (X+Y)} \|z\|$$

and $z \in (X + Y)$ if and only if there are $x \in X$ and $y \in Y$ such that $z = x + y$, so

$$\begin{aligned} p(X + Y) &= \inf_{x \in X, y \in Y} \|x + y\| \\ &\leq \inf_{x \in X, y \in Y} (\|x\| + \|y\|) \\ &= (\inf_{x \in X} \|x\|) + (\inf_{y \in Y} \|y\|) \\ &= p(X) + p(Y). \end{aligned}$$

Suppose $y \in \overline{F}$ and let Y be the coset to which it belongs. Then there is a sequence $(x_1, x_2, \ldots)$ in F such that

$$\lim_{n \to \infty} x_n = y.$$

Let

$$z_n = y - x_n.$$

Then

$$\lim_{n \to \infty} z_n = 0.$$

and

$$\lim_{n \to \infty} \|z_n\| = 0.$$

But $z_n \in Y$ for each n, so

$$\inf_{z \in Y} \|z\| \leq 0.$$

We already know it's greater than or equal to zero, so

$$\inf_{z \in Y} \|z\| = 0,$$

or $p(Y) = 0$. If p is a norm then $Y = 0$, so $y \in F$. This is true for all $y \in \overline{F}$, so $\overline{F} = F$. In other words, F is closed.

Suppose that $p(X) = 0$, i.e. that

$$\inf_{x \in X} \|x\| = 0.$$

Then there is a sequence $(x_1, x_2, \ldots)$ in X with

$$\lim_{n \to \infty} \|x_n\| = 0$$

and hence

$$\lim_{n \to \infty} x_n = 0.$$

If F is closed then so is X so $0 \in X$. But then $X = 0$. So if F is closed the p is a norm.

5. Let P be the subspace of $C([a, b])$ consisting of polynomials restricted to $[a, b]$. As usual, assume $a < b$. Show that P is a proper subspace, but not a closed subspace.

   Note: You may use, without proof, the Weierstrass approximation theorem.

   Solution: Since $a < b$, not every continuous function on $[a, b]$ is a polynomial. This doesn't really need a proof, but if you want one then observe the exp is continuous and that it cannot be equal in $[a, b]$ to a polynomial of degree d becuase its $d + 1$'st derivative has no zeroes. So $C([a, b])$ is a proper subspace.

   The Weierstrass approximation theorem says that for any $f \in C([a, b])$ and any $\epsilon > 0$ there is a $p \in P$ such that for all $t \in [a, b]$.

$$|f(t) - p(t)| < \epsilon$$

   In other words,

$$\|f - p\| < \epsilon$$

   where the norm is the usual one on $C([a, b])$. So f lies in the closure of P. Since f was arbitrary all of $C([a, b])$ lies in the closure of P. If P were closed then $C([a, b])$ would lie in P, but we've just seen that P is a proper subspace.