MA 3421 Assignment 2, Due 27 September 2018

Solutions

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1. Suppose E is a vector space and $p: E \to \mathbf{R}$ is a semi-norm. Let

$$F = \{x \in E : p(x) = 0\}.$$

Show that

(a) F is a subspace.

(b) there is a unique function $q: E/F \to \mathbf{R}$ such that $x \in X$ implies

$$q(X) = p(x).$$

(c) q is a norm.

Solution:

(a) If $\alpha, \beta \in \mathbf{K}$ and $x, y \in F$ then

$$p(\alpha x + \beta y) \leq p(\alpha x) + p(\beta y) = |\alpha| p(x) + |\beta| p(y) = 0,$$

by the various properties of seminorms and the definition of F. So $p(\alpha x + \beta y) \leq 0$. Using again the defining properties of siminorm,

$$p(\alpha x + \beta y) \geq 0.$$

It follows that

$$p(\alpha x + \beta y) = 0,$$

and hence that $\alpha x + \beta y \in F$.

(b) If $x, y \in X$ then, by the definition of a coset,

$$x - y \in F.$$

Then

$$p(x) = p(x - y + y) \leq p(x - y) + p(y) = p(y).$$

so $p(x) \leq p(y)$. Swapping the roles of x and y gives the reverse inequality and hence

$$p(x) = p(y).$$

So p is constant on cosets. We set q of each coset to be the value of p on that coset. That guarantees

$$q(X) = p(x).$$

when $x \in X$. If $r\colon E/F \to \mathbf{R}$ is another function with this property then, for each $X \in E/F$ choose an $x \in X$ and observe that

$$q(X) = p(x) = r(X).$$

Since this holds for all $X \in E/F$ we have $q = r$. In other words, the function q is unique.

(c) If $X, Y \in E/F$ then choose $x \in X$ and $y \in Y$. Then

$$q(X) = p(x) \geq 0,$$

$$q(\alpha X) = p(\alpha x) = |\alpha| p(x) = |\alpha| q(X),$$

and

$$q(X + Y) = p(x + y) \leq p(x) + p(y) = q(X) + q(Y),$$

so q is a semin-norm. We've used here the definition of addition and scalar muliplication in E/F to ensure that $\alpha x \in \alpha X$ and $x + y \in X + Y$. To see that q not just a semi-norm but actually a norm, note that if $q(X) = 0$ then there is an $x \in X$ with $p(x) = 0$. But then $x \in F$ and hence $X = F$. Since F is the zero element of the vectorspace E/F this means $X = 0$. So $q(X) = 0$ implies $X = 0$, as required.

2. Compute $\|A\|$ where $A \in \mathcal{L}(l^p(2))$, $p \geq 1$, and

$$A((\xi_1, \xi_2)) = (\xi_2, 0).$$

Compute $\|A^2\|$ and $\|A\|^2$.
Solution:

$$\|A(\xi_1, \xi_2)\| = \|(\xi_2, 0)\| = \left(|\xi_2|^p + |0|^p\right)^{1/p} \leq \left(|\xi_1|^p + |\xi_2|^p\right)^{1/p} = \|(\xi_1, \xi_2)\|$$

so 1 is a bound. If

$$(\xi_1, \xi_2) = (0, 1)$$

then

$$A(\xi_1, \xi_2) = (1, 0),$$

$$\|(\xi_1, \xi_2)\| = 1 = \|A(\xi_1, \xi_2)\|,$$

so 1 is the least bound, i.e.

$$\|A\| = 1.$$

Now

$$A^2(\xi_1, \xi_2) = A(\xi_2, 0) = (0, 0),$$

so $A^2 = 0$ and

$$\|A^2\| = 0,$$

while

$$\|A\|^2 = 1.$$

3. Show that the following analogues for normed spaces of familiar properties of the absolute value in $\mathbf{K}$, some of which have been already been used without comment in the notes and in lecture.

   (a) The norm is a continuous function.

   (b) $x_n \to y$ if and only if $\|x_n - y\| \to 0$.

   (c) Every Cauchy sequence, and hence every convergent sequence, is bounded.

   The proofs are in all cases the same as the proofs in $\mathbf{R}$, with absolute values replaced by norms where required.
   Solution:

   (a) Trivially

   $$\|x - y\| < \delta \implies \|x - y\| < \epsilon$$

   if $\delta = \epsilon$. Also trivially, if $\delta = \epsilon$ then $\delta > 0$ if $\epsilon > 0$. So there is, for every $y \in E$ and every $\epsilon > 0$ a $\delta > 0$ such that

   $$\|x - y\| < \delta \implies \|x - y\| < \epsilon.$$

   This is just the δ-ϵ definition of continuity of the norm function at the point y. Since y was arbitrary the norm function is continuous everywhere.

   (b) By definition, $\|x_n - y\| \to 0$ means there is an $N\colon \mathbf{R}^+ \to \mathbf{Z}^+$ such that

   $$j > N(\epsilon) \implies |\|x_n - y\| - 0| < \epsilon.$$

   This is obviously equivalent to

   $$j > N(\epsilon) \implies \|x_n - y\| < \epsilon,$$

   since the norm is non-negative. But this, in turn, is just the criterion for $x_n \to y$.

   (c) Suppose there is an $N\colon \mathbf{R}^+ \to \mathbf{Z}^+$ such that

   $$j, k > N(\epsilon) \implies \|x_j - x_k\| < \epsilon,$$

   as there must be if $(x_1, x_2, \ldots)$ is Cauchy. Choose $l > N(1)$, for example $l = N(1) + 1$. Then

   $$j > N(\epsilon) \implies \|x_j - x_l\| < 1,$$

By the triangle inequality,

$$\|x_j\| = \|x_j - x_l + x_l\| \leq \|x_j - x_l\| + \|x_l\|.$$

So

$$j > N(1) \Longrightarrow \|x_j\| < \|x_l\| + 1.$$

Let

$$\lambda = \max\left(\|x_1\|, \ldots, \|x_{N(1)}, \|x_l\| + 1\right).$$

Then, by the definition of the maximum,

$$j \leq N(1) \Longrightarrow \|x_j\| \leq \lambda.$$

But also

$$j > N(1) \Longrightarrow \|x_j\| \leq \lambda,$$

so

$$\|x_j\| \leq \lambda$$

for all j. In other words, $(x_1, x_2, \ldots)$ is bounded.

4. Show that if

$$\lim_{j\to\infty} A_j = C$$

in $\mathcal{L}(F, G)$ and

$$\lim_{j\to\infty} B_j = D$$

in $\mathcal{L}(E, F)$ then

$$\lim_{j\to\infty} A_j B_j = CD$$

in $\mathcal{L}(E, G)$.
Solution: By hypothesis there are $N_A, N_B \colon \mathbf{R}^+ \to \mathbf{Z}^+$ such that

$$j > N_A(\epsilon) \Longrightarrow \|A_j - C\| < \epsilon$$

and

$$j > N_B(\epsilon) \Longrightarrow \|B_j - D\| < \epsilon.$$

From the previous problem we know that A_j is bounded,

$$\|A_j\| \leq \lambda$$

for some $\lambda \geq 0$. Choose $\mu > \lambda$ and $\nu > \|D\|$ and set

$$M(\epsilon) = \max\left(N_A\left(\frac{\epsilon}{2\nu}\right) N_B\left(\frac{\epsilon}{2\mu}\right)\right)$$

for $\epsilon > 0$. Note that

$$A_j B_j - CD = A_j(B_j - D) + (A_j - C)D$$

so

$$\|A_j B_j - CD\| \leq \|A_j\| \|B_j - D\| + \|A_j - C\| \|D\|.$$

If $j > M(\epsilon)$ then $j > N_A(\epsilon/2\nu)$, so

$$\|A_j - C\| \|D\| < \frac{\epsilon}{2\nu} \nu = \frac{\epsilon}{2},$$

and $j > N_B(\epsilon/2\mu)$, so

$$\|A_j\| \|B_j - D\| < \mu \frac{\epsilon}{2\mu} = \frac{\epsilon}{2},$$

and hence

$$\|A_j B_j - CD\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

So there is an $M \colon \mathbf{R}^+ \to \mathbf{Z}^+$ such that

$$j > M(\epsilon) \Longrightarrow \|A_j B_j - CD\| < \epsilon.$$

In other words,

$$\lim_{j \to \infty} A_j B_j = CD.$$