MA 3421 Assignment 1, Due 20 September 2018

Solutions

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1. Suppose that $f\colon E \to F$ is a function from one metric space to another. Show that the following statements are equivalent:

   (a) If $x_n \to y$ in E then $f(x_n) \to f(y)$ in F.

   (b) For all $\epsilon > 0$ and $y \in E$ there is a $\delta > 0$ such that, for all $x \in E$,
   $$\|x - y\| < \delta$$
   implies
   $$\|f(x) - f(y)\| < \epsilon.$$

   Solution: Suppose first that for all $\epsilon > 0$ and $y \in E$ there is a $\delta > 0$ such that, for all $x \in E$, $\|x - y\| < \delta$ implies $\|f(x) - f(y)\| < \epsilon$. If $x_n \to y$ in E then there is an $N \in \mathbf{Z}^+$ such that $j > N$ implies $|x_j - y| < \delta$ and hence $|f(x_j) - f(y)| < \epsilon$. Since there is such an N for every $\epsilon > 0$ it follows that $f(x_n) \to f(y)$ in F.

   Suppose now this is not true. In other words, suppose that for some $\epsilon > 0$ and $y \in E$ there is no $\delta > 0$ for which $\|x-y\| < \delta$ implies $\|f(x)-f(y)\| < \epsilon$. This means that for every $\delta > 0$ there is an $x \in E$ with $\|x - y\| < \delta$ but $\|f(x) - f(y)\| \geq \epsilon$. This is true in particular then for $\delta = 1/n$. Let x_n be the corresponding n. Then $x_n \to y$ because $\|x_n - y\| < 1/n$. But $\|f(x_n) - f(y)\| \geq \epsilon > 0$ for all n, and this ϵ is independent of n, so $f(x_n) \not\to f(y)$. This shows that it is not true that if $x_n \to y$ in E then $f(x_n) \to f(y)$ in F.

2. Show that convergence in (s) and pointwise convergence are equivalent.
   Solution: Suppose that $x_j \to y$ in (s) and denote the elements of x_j by $x_l = (\xi_1^{(j)}, \xi_2^{(j)}, \ldots)$ and the elements of y by $y = (\eta_1, \eta_2, \ldots)$. If $\epsilon > 0$ then let
   $$\epsilon' = \frac{1}{2^m} \frac{\epsilon}{1 + \epsilon}$$

and observe that $\epsilon' > 0$ so, by the definition of convergence is an $N \in \mathbf{Z}^+$ such that $j > N$ implies

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \frac{|\xi_n^{(j)} - \eta_n|}{1 + |\xi_n^{(j)} - \eta_n|} < \epsilon'.$$

For any $m \in \mathbf{Z}^+$ then

$$\frac{|\xi_m^{(j)} - \eta_m|}{1 + |\xi_m^{(j)} - \eta_m|} < 2^m \epsilon',$$

and hence

$$|\xi_m^{(j)} - \eta_m| < \frac{2^m \epsilon'}{1 - 2^m \epsilon'} = \epsilon.$$

So for any $\epsilon > 0$ there is an N such that for $j > N$

$$|\xi_m^{(j)} - \eta_m| < \epsilon.$$

In other words,

$$\lim_{j \to \infty} \xi_m^{(j)} = \eta_m,$$

so $x_j \to y$ pointwise.

Conversely, suppose that $x_j \to y$ pointwise. Suppose that $\epsilon > 0$, Since

$$\lim_{M \to \infty} \frac{1}{2^M} = 0$$

there is, for each $\epsilon > 0$, an M such that

$$2^M < \epsilon.$$

Let

$$\epsilon' = \epsilon - \frac{1}{2^M}.$$

Then $\epsilon' > 0$ For each $j \leq M$ we have

$$\lim_{j \to \infty} \xi_m^{(j)} = \eta_m,$$

so, by the definition of convergence in $\mathbf{K}$, there is, for each $j \leq M$, an $N_j \in \mathbf{Z}^+$ such that if $k > N_j$ then

$$|\xi_k^{(j)} - \eta_k| < \epsilon'.$$

Set

$$N = \max_{1 \leq j \leq M} N_j.$$

If $k > N$ then $k > N_j$ for each $j \leq M$,

$$\frac{|\xi_k^{(j)} - \eta_k|}{1 + |\xi_k^{(j)} - \eta_k|} < |\xi_k^{(j)} - \eta_k| < \epsilon'$$

and hence

$$\sum_{j=1}^{M} \frac{1}{2^j} \frac{|\xi_k^{(j)} - \eta_k|}{1 + |\xi_k^{(j)} - \eta_k|} < \sum_{j=1}^{M} \frac{1}{2^j} \epsilon' = \left(1 - \frac{1}{2^M}\right) \epsilon' < \epsilon'.$$

For $j > M$ we have

$$\frac{|\xi_m^{(j)} - \eta_m|}{1 + |\xi_m^{(j)} - \eta_m|} < 1$$

and hence

$$\sum_{j=M+1}^{\infty} \frac{1}{2^j} \frac{|\xi_k^{(j)} - \eta_k|}{1 + |\xi_k^{(j)} - \eta_k|} < \sum_{j=M+1}^{\infty} \frac{1}{2^j} = \frac{1}{2^M}.$$

Combining these,

$$\sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|\xi_k^{(j)} - \eta_k|}{1 + |\xi_k^{(j)} - \eta_k|} < \epsilon' + \frac{1}{2^M} = \epsilon.$$

So there is for each $\epsilon > 0$, an N such that if $j > N$ then

$$d(x_j, y) < \epsilon,$$

where the metric d is the one on (s). In other words,

$$\lim_{j \to \infty} x_j = y$$

in (s).

3. Show that convergence in $C([a, b])$ and pointwise convergence are *not* equivalent for $a < b$.
Solution: It suffices to exhibit a sequence $(x_1, x_2, \ldots)$ in $C([a, b])$ which converges pointwise to a $y \in C([a, b])$, but which does not converge in $C([a, b])$. There are plenty of such sequences, one of which is

$$x_j(t) = \begin{cases} (j+2)\frac{t-a}{b-a} & \text{if } a \leq t \leq a + s_j, \\ (j+2)\frac{a+2s_j-t}{b-a}. & \text{if } a + s_j \leq t \leq a + 2s_j, \\ 0 & \text{if } a + 2s_j \leq t \leq b. \end{cases}$$

where

$$s_j = \frac{b-a}{j+2}.$$

Clearly $a < a + s_j < a + 2s_j < b$. The definition is consistent because

$$x_j(a + s_j) = 1, \quad x_j(a + 2s_j) = 0$$

no matter which of the two available expressions one uses to evaluate them. Now $x_j(a) = 0$ for all j, so clearly

$$\lim_{j\to\infty} x_j(t) = 0$$

when $t = a$. For any $t > a$ we have

$$x_j(t) = 0$$

for

$$j \geq 2\frac{b - a}{t - a} - 2,$$

since t satisfies $a + 2s_j \leq t \leq b$ for such j. It follows that

$$\lim_{j\to\infty} x_j(t) = 0$$

for all $t \in [a, b]$. In other words $x_j \to 0$ pointwise. But it follows from

$$x_j(a + s_j) = 1$$

that

$$d(x_j, 0) \geq 1.$$

The metric here is, of course, the one on $C([a, b])$. In fact $d(x_j, 0) = 1$, but the inequality already suffices to show that $x_j \not\to 0$ in $C([a, b])$.

4. Suppose that E is equipped with the discrete metric. Show that all subsets are both open and closed and that a subset is compact if and only if it is finite.
Solution: Suppose $S \subset E$ and $x \in S$ then there is a $\rho > 0$ such that, for all $y \in E$, $d(x, y) < r$ implies $y \in S$. In fact any $\rho \in (0, 1)$ will accomplish this, because then $d(x, y) < \rho$ implies $x = y$. But this is just then definition of an open set. So any $S \subset E$ is open. But then $E - S$ is also open, since it's also a subset of E, so S is closed.

   Finite subsets are compact. Indeed, if $S \subset E$ is finite then every sequence in S has a constant subsequence, and constant sequences always converge. If S is not finite then we can choose an infinite sequence $(s_1, s_2, \ldots)$ of distinct elements of S. This sequence is bounded, because $d(s_j, s_k) \leq 1$ for all j, k, but no subsequence converges. To see this note that all elements of the subsequence are also distinct and hence have distance 1, so the Cauchy criterion fails for any $\epsilon \in (0, 10$.

5. Are the sequences $e_1, e_2, \ldots$ a basis for the sequence space (s)? Recall that e_j is the sequence whose j'th element is 1 and all other elements are 0.
Solution: No. The sequence $(1, 1/2, 1/3, \ldots)$, for example, is not a finite linear combination of the e's, so they clearly don't span.

6. Show that the quotient of (c) by its subspace (c_0) is one-dimensional.
Solution: Denote the quotient space by V. Let

$$z = (\zeta_1, \zeta_2, \zeta_3, \ldots) = (1, 1, 1, \ldots).$$

Clearly $z \in (c)$. V is, by definition, the set of c_0-cosets in (c). Let Z be the coset to which z belongs. If $\alpha \in K$ and $x = (\xi_1, \xi_2, \ldots) \in \alpha Z$ then, by definition,

$$x - \alpha z \in c_0$$

so

$$\lim_{n\to\infty} (\xi_n - \alpha\zeta_n) = 0,$$

from which it follows that

$$\lim_{n\to\infty} \xi_n = \alpha \lim_{n\to\infty} \zeta_n + \lim_{n\to\infty} (\xi_n - \alpha\zeta_n) = \alpha.$$

This belongs to (c_0) if and only if $\alpha = 0$. (c_0) is the 0 element of V, so what we've shown is that $\alpha Z = 0$ in V if and only if $\alpha = 0$. The 'only if' part of this statement expresses the linear independence of the subset $B = \{Z\}$ of V. For any $X \in V$, choose an $x \in X$ and let

$$\alpha = \lim_{n\to\infty} \xi_n.$$

Then

$$\lim_{n\to\infty} (\xi_n - \alpha\zeta_n) = 0,$$

so $x - \alpha z \in (c_0)$ and hence $X = \alpha Z$ in V. Since $X \in V$ was arbitrary it follows that any element of V is a linear combination of elements of B. In other words, B spans V. Since we already know it is linearly independent it follows that B is a basis for V. Clearly it has one element, so the dimension of V is one.