MA 3421 Assignment 8, Due 22 November 2018

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1. The norm version of the Hahn-Banach theorem was proved in lecture and in the notes using Zorn's Lemma. Give a proof without using Zorn's Lemma in the special case where the Banach space is a Hilbert space.

2. Show that $\mathbf{R} - \mathbf{Q}$ is not a countable union of closed sets.
   Hint: Use the Baire Category Theorem.

3. Suppose that $A\colon H \to H$ is linear, where H is a Hilbert space, and that

   $$(Ax|y) = (x|Ay)$$

   for all A. Show that A is symmetric.
   Hint: The problem is to show boundedness, since all the other properties of symmetric operators are true by assumption. Use the Closed Graph Theorem.

4. Suppose that $T_{j,k} \in \mathcal{L}(E, F)$ for each $j, k \in \mathbf{Z}^+$, where E and F are Banach spaces. It's trivially true that if there is an $x \in E$ such that for each $j \in \mathbf{Z}^+$, $T_{j,k}x$ is unbounded as a function of k then for each $j \in \mathbf{Z}^+$ there is an $x \in E$ such that $T_{j,k}$ is unbounded as a function of k. The converse is called the Principle of Condensation of Singularities: If for each $j \in \mathbf{Z}^+$ there is an $x \in E$ such that $T_{j,k}$ is unbounded as a function of k then there is an $x \in E$ such that for each $j \in \mathbf{Z}^+$, $T_{j,k}x$ is unbounded as a function of k. Prove it.
   Hint: Consider

   $$C_{j,k,l} = \{x \in E\colon \|T_{j,k}x\| \le l\}.$$

   Is

   $$\cup_{j\in\mathbf{Z}^+} \cup_{l\in\mathbf{Z}^+} \cap_{k\in\mathbf{Z}^+} C_{j,k,l} = E?$$

   Use the Baire Category Theorem. You'll face the same problem that we faced in the proof of the Uniform Boundedness Theorem, that the open ball need not be centred at the origin, but you can solve it in the same way we did there.

5. We say that a sequence $(A_1, A_2, \ldots)$ in $\mathcal{L}(E, F)$ *converges strongly* to $C \in \mathcal{L}(E, F)$ if for all $x \in$ the sequence $(A_1x, A_2x, \ldots)$ in F converges to Cx.

(a) Show that if $(A_1, A_2, \ldots)$ converges to C then it converges strongly.

(b) Given an example of E, F, $(A_1, A_2, \ldots)$ and C for which $(A_1, A_2, \ldots)$ converges strongly to C but does not converge to C.[1]

Hint: The first part is easy. For the second part you might find it helpful to revise the discussion of weak convergence.

6. Suppose that E, F and G are Banach spaces, that $(A_1, A_2, \ldots)$ is a sequence in $\mathcal{L}(F, G)$ which converges strongly to C and that $(B_1, B_2, \ldots)$ is a sequence in $\mathcal{L}(E, F)$ which converges strongly to D. Show that that the sequence $(A_1B_1, A_2B_2, \ldots)$ in $\mathcal{L}(E, G)$ converges strongly to CD.
Hint: Use the Uniform Boundedness Principle.

[1] Yes, the terminology is horrible. You might reasonably expect based on the rules of English grammar that strong convergence is a stronger condition than ordinary convergence, but it's actually a weaker condition.