MA 3421 Assignment 7, Due 15 November 2018

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1. Prove the following 'converse' to the spectral theorem:
   If $\{u_1, u_2, \ldots\}$ is an orthonormal set in a Hilbert space E and $(\lambda_1, \lambda_2, \ldots)$ is a sequence in $\mathbf{R}$ with limit 0 and $|\lambda_1| \geq |\lambda_2| \geq \cdots$ then
   $$Ax = \sum_{j=1}^{\infty} \lambda_j \, (x|u_j) \, u_j$$
   defines a compact symmetric operator on E.
   Hint: Remember that the limit of a sequence of compact operators is compact.

2. Show that every compact symmetric operator A on a Hilbert space E can be written in the form
   $$A = B - C$$
   where B and C are compact positive operators.

3. The left and right unilateral shifts $L, R \in \mathcal{L}(l^2)$ are defined by
   $$Lx = (\xi_2, \xi_3, \ldots), \quad Rx = (0, \xi_1, \xi_2, \ldots),$$
   where
   $$x = (\xi_1, \xi_2, \ldots).$$
   (a) Show that neither L nor R is compact.
   (b) Show that neither L nor R is symmetric.
   (c) Find the eigenvectors and eigenvalues of L.
   (d) Find the eigenvectors and eigenvalues of R.

4. Show that $\{1, \sqrt{2}\cos(\pi t), \sqrt{2}\cos(2\pi t), \ldots\}$ and $\{\sqrt{2}\sin(\pi t), \sqrt{2}\sin(2\pi t), \ldots\}$ are orthonormal bases for $L^2([0, 1])$.
   Hint: Write them as solutions to a Sturm-Liouville problem.

5. Show that $\{\frac{1}{\sqrt{2}}, \cos(\pi t), \sin(\pi t), \cos(2\pi t), \sin(2\pi t), \ldots\}$ is an orthonormal basis for $L^2([-1,1])$.
Hint: Use the result of the previous problem. You may find the functions $A\colon L^2([-1,1]) \to L^2([0,1]) \oplus L^2([0,1])$ and $B\colon L^2([0,1]) \oplus L^2([0,1]) \to L^2([-1,1])$ useful where

$$A(x) = (y, z), \quad y(t) = \frac{x(t) + x(-t)}{\sqrt{2}}, \quad z(t) = \frac{x(t) - x(-t)}{\sqrt{2}}$$

and

$$B((y, z)) = x, \quad x(t) = \begin{cases} \frac{y(t)+z(t)}{\sqrt{2}} & \text{if } t > 0, \\ \frac{y(t)-z(t)}{\sqrt{2}} & \text{if } t < 0. \end{cases}$$