

MA 3421 Assignment 6, Due 8 November 2018

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1. Suppose S is a set, not necessarily finite or countable, $f: S \rightarrow \mathbf{R}$ is a function, $\mu, \nu \in \mathbf{R}$ and, for all finite subsets $\subseteq S$ we have

$$\mu \leq \sum_{u \in S} f(u) \leq \nu.$$

Show that the set

$$N = \{u \in S: f(u) \neq 0\}$$

is at most countable.

Note: This is a generalisation of an argument given both in the notes and in lecture.

2. The dual to a normed space E is, by definition, $E' = \mathcal{L}(E, \mathbf{K})$. This is a normed space, because \mathbf{K} is a normed space and $\mathcal{L}(E, F)$ is a normed space for any normed spaces E and F . The double dual is the dual of the dual, $E'' = \mathcal{L}(E', \mathbf{K}) = \mathcal{L}(\mathcal{L}(E, \mathbf{K}), \mathbf{K})$. Show that E' and E'' are in fact Banach spaces.

3. Define $J(x)$, a function from E' to \mathbf{K} , by

$$(J(x))(f) = f(x)$$

for all $x \in E$, $f \in E'$. Show the following:

- (a) $J(x): E' \rightarrow \mathbf{K}$ is linear and bounded, so $J(x) \in E''$.
 - (b) $J: E \rightarrow E''$ is linear and bounded, so $J \in \mathcal{L}(E, E'')$.
 - (c) If E is an inner product space then J is injective.
4. If E is a Hilbert space then it's a normed space and so E' is a Banach space. Show that it is, in fact, a Hilbert space, i.e. that the norm on E' is induced by an inner product on E' .

Note: Use the Riesz Representation Theorem.

5. Suppose that E and F are normed spaces and $A: E \rightarrow F$ linear. and that the image of every bounded sequence in E has a convergent subsequence, i.e. that for every bounded sequence (x_1, x_2, \dots) in E the sequence (Ax_1, Ax_2, \dots) in F has a convergent subsequence. Show that A is compact.

Note: Your first task is to figure out why this is not just the definition of compactness.

6. Suppose $s \in \mathbf{K}$ and $\operatorname{Re} s \geq 0$. If $x = (\xi_1, \xi_2, \dots) \in l^2$ then

$$A_n x = (1^{-s} \xi_1, 2^{-s}, \dots, n^{-s} \xi_n, 0, 0, \dots),$$

$$Bx = (1^{-s} \xi_1, 2^{-s}, \dots, n^{-s} \xi_n, (n+1)^{-s} \xi_{n+1}, \dots),$$

belong to l^2 and define operators $A_n, B \in \mathcal{L}(l^2)$.

- (a) Show that

$$\lim_{n \rightarrow \infty} A_n = B$$

if $\operatorname{Re} s > 0$.

- (b) Show that A_n is compact if $\operatorname{Re} s \geq 0$.

Hint: Write A_n as a product of operators with finite dimensional domain or range.

- (c) Show that B is compact if $\operatorname{Re} s > 0$.

- (d) Show that B is symmetric if and only if s is real.