MA 3421 Assignment 5, Due 1 November 2018

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1. It was shown in lecture that if $x_1, x_2, \ldots$ are pairwise orthogonal in an inner product space E and
$$\sum_{j=1}^{\infty} x_j$$
converges then
$$\left\| \sum_{j=1}^{\infty} x_j \right\|^2 = \sum_{j=1}^{\infty} \|x_j\|^2.$$
Show that if $x_1, x_2, \ldots$ are pairwise orthogonal in a Hilbert space E and
$$\sum_{j=1}^{\infty} \|x_j\|^2$$
converges then
$$\sum_{j=1}^{\infty} x_j$$
converges and
$$\left\| \sum_{j=1}^{\infty} x_j \right\|^2 = \sum_{j=1}^{\infty} \|x_j\|^2.$$

2. The functions
$$x_j(t) = t^j \exp(-t^2/2)$$
in $L^2(\mathbf{R})$ are linearly independent. Use them to find an orthonormal set via Gram-Schmidt.
Note: There are, of course, infinitely many of them, so you're not going to be able to list them all. Just find the ones of degree less than 4.

3. Suppose $\{x_1, \ldots, x_n\} \subset E$ where E is an inner product space and let A be the matrix whose j'th row, k'th column is
$$\alpha_{j,k} = (x_j | x_k) .$$

(a) Show that A is Hermitian and positive semi-definite.

(b) Show that $\{x_1, \ldots, x_n\}$ is linearly dependent if and only if $\det A = 0$.

4. Suppose that $P \in \mathcal{L}(E)$,

$$P^2 = P$$

and, for all $x, y \in E$,

$$(Px|y) = (x|Py)\,.$$

Show that there is a closed subspace F such that P is the orthogonal projection onto F.

5. The proof given in the notes that any orthonormal set is a subset of a maximal orthonormal set uses Zorn's Lemma and so is non-constructive. Give a constructive proof in $l^2(n)$, i.e. describe an algorithm which, given an orthonormal set $\{u_1, \ldots, u_k\} \subseteq l^2$, constructs $u_{k+1}, \ldots, u_n$ such that $\{u_1, \ldots, u_n\}$ is an orthonormal set.
Note: You can assume for purposes of this problem that you have a way of doing exact arithmetic in $\mathbf{K}$. If you're feeling adventurous you can try to give an algorithm which has some hope of working in real life.
Hint: Try to adapt the Gram-Schmidt algorithm.