

MAU23205 2021-2022 Practice Problem Set 5  
*Solutions*

1. Let

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For each of the following  $A$  find a symmetric  $C$  such that  $A^T B + BA + C = O$ . For which  $A$ 's is the  $B$  you found positive definite?

(a)

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

*Solution:*

$$B = \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$

is positive definite.

(b)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

*Solution:*

$$B = \begin{bmatrix} 23/20 & -11/20 \\ -11/20 & 3/20 \end{bmatrix}$$

is not positive definite.

(c)

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

*Solution:*

$$B = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

is not positive definite.

2. Find the Green's function for the second order scalar equation

$$x''(t) + 2x'(t) + 2x(t) = 0.$$

*Solution:* The Green's function for an  $m \times m$  matrix was defined in Lecture 18 to be  $G(t, s) = w_{1,m}(t, s)$  where  $W$  is the fundamental matrix. This equation is linear constant coefficient so the fundamental matrix is

$$W(t, s) = \exp((t - s)A)$$

where  $A$  is the coefficient matrix of the associated first order system:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}.$$

The matrix exponential is

$$\exp((t-s)A) = \exp(-(t-s)) \begin{bmatrix} \cos(t-s) + \sin(t-s) & \sin(t-s) \\ -2\sin(t-s) & \cos(t-s) - \sin(t-s) \end{bmatrix}$$

so the Green's function is

$$G(t, s) = w_{1,2}(t, s) = \exp(-(t-s)) \sin(t-s).$$

3. Find the fundamental matrix  $W$  for

$$A(t) = \begin{bmatrix} 1/t & t \\ 0 & 1 \end{bmatrix}$$

*Solution:* This is a partially decoupled system as in Lecture 19.

$$w_{1,1}(t, r) = \exp\left(\int_r^t a_{1,1}(s) ds\right) = \exp\left(\int_r^t \frac{1}{s} ds\right) = \log(t/r),$$

$$w_{2,2}(t, r) = \exp\left(\int_r^t a_{2,2}(s) ds\right) = \exp\left(\int_r^t ds\right) = t - r,$$

$$\begin{aligned} w_{1,2}(t, r) &= \int_r^t w_{1,1}(t, s) a_{1,2}(s) w_{2,2}(s, r) ds = \int_r^t \log(t/s) s(s-r) ds \\ &= \left[ \frac{1}{3} s^3 \log(t/s) + \frac{1}{9} s^3 - \frac{1}{2} r s^2 \log(t/s) - \frac{1}{4} r s^2 \right]_{s=r}^{s=t} \\ &= \frac{1}{9} t^3 - \frac{1}{4} r t^2 + \frac{1}{6} r^3 \log(t/r) + \frac{5}{36} r^3 \\ w_{2,1}(t, r) &= 0. \end{aligned}$$

4. The Bessel equation of order  $\nu$  is

$$x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y(x) = 0.$$

Show that there is no non-zero power series solution unless  $\nu$  is an integer, in which case there is one whose first non-zero term is the  $x^{|\nu|}$  term. Where does it converge?

*Solution:* We look for a solution of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^j.$$

Then

$$y'(x) = \sum_{j=0}^{\infty} j a_j x^{j-1}$$

and

$$y''(x) = \sum_{j=0}^{\infty} (j-1) j a_j x^{j-2}$$

so

$$x^2 y(x) = \sum_{j=0}^{\infty} a_j x^{j+2},$$

$$x y'(x) = \sum_{j=0}^{\infty} j a_j x^j,$$

and

$$x^2 y''(x) = \sum_{j=0}^{\infty} (j-1) j a_j x^j.$$

It follows that

$$x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y(x) = 0 = \sum_{j=0}^{\infty} (j^2 - \nu^2) a_j x^j + \sum_{j=0}^{\infty} a_j x^{j+2}.$$

We can reindex the second sum to get

$$x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y(x) = 0 = \sum_{j=0}^{\infty} (j^2 - \nu^2) a_j x^j + \sum_{j=2}^{\infty} a_{j-2} x^j.$$

This will be zero if and only if the coefficient of  $x^j$  is zero for each  $j$ . The first two coefficients are  $-\nu^2 a_0$  and  $(1 - \nu^2) a_1$ . We need these to be zero. If  $\nu = 0$  then  $a_0$  is arbitrary but  $a_1 = 0$ . If  $\nu = \pm 1$  then  $a_0 = 0$  any  $a_1$  is arbitrary. For any other value of  $\nu$  both  $a_0$  and  $a_1$  must be zero. For  $j \geq 2$  the coefficient is  $(j^2 - \nu^2) a_j + a_{j-2}$  which is zero if and only if

$$a_j = -\frac{a_{j-2}}{j^2 - \nu^2},$$

at least if  $j \neq \pm \nu$ . If  $\nu$  is not any integer then  $j$  is never  $\pm \nu$  so from the fact that  $a_0 = 0$  and  $a_1 = 0$  it follows that all the remaining coefficients are zero. If  $\nu$  is an integer then we can take  $a_j = 0$  for  $j < |\nu|$  and  $a_{|\nu|}$  can be chosen arbitrarily since  $(j^2 - \nu^2) a_j + a_{j-2}$  will still be zero for  $j = |\nu|$ . The equation

$$a_j = -\frac{a_{j-2}}{j^2 - \nu^2},$$

then determines  $a_{|\nu|+2}$ ,  $a_{|\nu|+4}$ , etc. The coefficients  $a_{|\nu|+1}$ ,  $a_{|\nu|+3}$ , etc are all zero because  $a_{|\nu|-1}$  was zero. The ratio of successive non-zero terms is

$$\frac{a_j x^j}{a_{j-2} x^{j-2}} = -\frac{x^2}{j^2 - \nu^2}$$

which tends to zero as  $j$  tends to infinity for any value of  $x$  so this solution converges everywhere.