MAU23205 2021-2022 Practice Problem Set 5

Solutions

1. Let

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

For each of the following A find a symmetric C such that $A^T B + BA + C = O$. For which A's is the B you found positive definite?

(a)

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$$

Solution:

$$B = \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$$

is positive definite.

(b)

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Solution:

$$B = \begin{bmatrix} 23/20 & -11/20 \\ -11/20 & 3/20 \end{bmatrix}$$

is not positive definite.

(c)

$$A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

Solution:

$$B = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$$

is not positive definite.

2. Find the Green's function for the second order scalar equation

$$x''(t) + 2x'(t) + 2x(t) = 0.$$

Solution: The Green's function for an $m \times m$ matrix was defined in Lecture 18 to be $G(t,s) = w_{1,m}(t,s)$ where W is the fundamental matrix. This equation is linear constant coefficient so the fundamental matrix is

$$W(t,s) = \exp((t-s)A)$$

where A is the coefficient matrix of the associated first order system:

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}.$$

The matrix exponential is

$$\exp((t-s)A) = \exp(-(t-s)) \begin{bmatrix} \cos(t-s) + \sin(t-s) & \sin(t-s) \\ -2\sin(t,s) & \cos(t-s) - \sin(t-s) \end{bmatrix}$$

so the Green's function is

$$G(t,s) = w_{1,2}(t,s) = \exp(-(t-s))\sin(t-s).$$

3. Find the fundamental matrix W for

$$A(t) = \begin{bmatrix} 1/t & t \\ 0 & 1 \end{bmatrix}$$

Solution: This is a partially decoupled system as in Lecture 19.

$$w_{1,1}(t,r) = \exp\left(\int_r^t a_{1,1}(s)\,ds\right) = \exp\left(\int_r^t \frac{1}{s}\,ds\right) = \log(t/r),$$

$$w_{2,2}(t,r) = \exp\left(\int_r^t a_{2,2}(s)\,ds\right) = \exp\left(\int_r^t ds\right) = t - r,$$

$$\begin{aligned} w_{1,2}(t,r) &= \int_r^t w_{1,1}(t,s)a_{1,2}(s)w_{2,2}(s,r)\,ds = \int_r^t \log(t/s)s(s-r)\,ds \\ &= \left[\frac{1}{3}s^3\log(t/s) + \frac{1}{9}s^3 - \frac{1}{2}rs^2\log(t/s) - \frac{1}{4}rs^2\right]_{s=r}^{s=t} \\ &= \frac{1}{9}t^3 - \frac{1}{4}rt^2 + \frac{1}{6}r^3\log(t/r) + \frac{5}{36}r^3 \\ w_{2,1}(t,r) &= 0. \end{aligned}$$

4. The Bessel equation of order ν is

$$x^2y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0.$$

Show that there is no non-zero power series solution unless ν is an integer, in which case there is one whose first non-zero term is the $x^{|\nu|}$ term. Where does it converge?
Solution: We look for a solution of the form

$$y(x) = \sum_{j=0}^{\infty} a_j x^j.$$

Then

$$y'(x) = \sum_{j=0}^{\infty} j a_j x^{j-1}$$

and

$$y''(x) = \sum_{j=0}^{\infty} (j-1) j a_j x^{j-2}$$

so

$$x^2 y(x) = \sum_{j=0}^{\infty} a_j x^{j+2},$$

$$x y'(x) = \sum_{j=0}^{\infty} j a_j x^j,$$

and

$$x^2 y''(x) = \sum_{j=0}^{\infty} (j-1) j a_j x^j.$$

It follows that

$$x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y(x) = 0 = \sum_{j=0}^{\infty} (j^2 - \nu^2) a_j x^j + \sum_{j=0}^{\infty} a_j x^{j+2}.$$

We can reindex the second sum to get

$$x^2 y''(x) + x y'(x) + (x^2 - \nu^2) y(x) = 0 = \sum_{j=0}^{\infty} (j^2 - \nu^2) a_j x^j + \sum_{j=2}^{\infty} a_{j-2} x^j.$$

This will be zero if and only if the coefficient of x^j is zero for each j. The first two coefficients are $-\nu^2 a_0$ and $(1-\nu^2)a_1$. We need these to be zero. If $\nu = 0$ then a_0 is arbitrary but $a_1 = 0$ If $\nu = \pm 1$ then $a_0 = 0$ any a_1 is arbitrary. For any other value of ν both a_0 and a_1 must be zero. For $j \geq 2$ the coefficient is $(j^2 - \nu^2)a_j + a_{j-2}$ which is zero if and only if

$$a_j = -\frac{a_{j-2}}{j^2 - \nu^2},$$

at least if $j \neq \pm\nu$. If ν is not any integer then j is never $\pm\nu$ so from the fact that $a_0 = 0$ and $a_1 = 0$ it follows that all the remaining coefficients are zero. If ν is an integer then we can take $a_j = 0$ for $j < |\nu|$ and $a_{|\nu|}$ can be chosen arbitrarily since $(j^2 - \nu^2)a_j + a_{j-2}$ will still be zero for $j = |\nu|$. The equation

$$a_j = -\frac{a_{j-2}}{j^2 - \nu^2},$$

then determines $a_{|\nu|+2}$, $a_{|\nu|+4}$, etc. The coefficents $a_{|\nu|+1}$, $a_{|\nu|+3}$, etc are all zero because $a_{|\nu|-1}$ was zero. The ratio of successive non-zero terms is

$$\frac{a_j x^j}{a_{j-2} x^{j-2}} = -\frac{x^2}{j^2 - \nu^2}$$

which tends to zero as j tends to infinity for any value of x so this solution converges everywhere.