MAU23205 2021-2022 Practice Problem Set 4

1. (a) Suppose
$$a(z) = \sum_{i=0}^{m} \alpha_i z^i$$
and
$$b(z) = \sum_{j=0}^{n} \beta_j z^j.$$
Show that
$$a(z)b(z) = \sum_{k=0}^{m+n} \gamma_k z^k$$
where
$$\gamma_k = \sum_{i+j=k} \alpha_i \beta_j.$$

(b) Suppose
$$\sum_{i=0}^{m} \alpha_i x^{(i)}(t) = f(t)$$
and
$$\sum_{j=0}^{n} \beta_j f^{(j)}(t) = 0.$$
Show that
$$\sum_{k=0}^{m+n} \gamma_k x^{(k)}(t) = 0,$$
where
$$\gamma_k = \sum_{i+j=k} \alpha_i \beta_j.$$

(c) Suppose
$$\sum_{i=0}^{m} \alpha_i x^{(i)}(t) = f(t)$$
and
$$\sum_{j=0}^{n} \beta_j f^{(j)}(t) = 0.$$
Let
$$a(z) = \sum_{i=0}^{m} \alpha_i z^i$$
and
$$b(z) = \sum_{j=0}^{n} \beta_j z^j.$$

Show that x is a linear combination of basic solutions $q_{\lambda,h}$, $r_{\kappa,\omega,h}$ and $s_{\kappa,\omega,h}$ where λ is a real root of a or b, $\kappa \pm i\omega$ is a pair of complex conjugate complex roots of a or b, and h is less than the sum of the multiplicities of λ or $\kappa \pm i\omega$ as roots of a and b.
Note: When computing the sum of multiplicities we regard non-roots as having multiplicity zero. So if λ is a root of a but not of b then we take the multiplicity of λ as a root of a, if λ is a root of b but not of a then we take the multiplicity of λ as a root of b, and if λ is a root of both a and b then we add the multiplicities.

2. Solve the initial value problem

$$x(0) = x_0, \qquad x'(0) = y_0$$

for the differential equation

$$x''(t) + 2x'(t) + 2x(t) = \exp(-t)\cos(t).$$

Hint: Use the result of the previous problem, noting that if $f(t) = \exp(-t)\cos(t)$ then $f''(t) + 2f'(t) + 2f(t) = 0$.

3. Compute $\exp(tA)$ where

(a)

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

(b)

$$A = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 1 & -1 \\ -4 & 6 & -2 \end{bmatrix}$$